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Algebraic operator method for the construction of solitary solutions to nonlinear differential equations

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ABSTRACT

Solutions of the KdV equation are derived by the algebraic operator method based on generalized operators of differentiation. The algebraic operator method based on the generalized operator of differentiation is exploited for the derivation of analytic solutions to the KdV equation. The structure of solitary solutions and explicit conditions of existence of these solutions in the subspace of initial conditions are derived. It is shown that special solitary solutions exist only on a line in the parameter plane of initial and boundary conditions. This new theoretical result may lead to important findings in a variety of practical applications.

1. Introduction

The Korteweg-de Vries equation takes the form

$$\frac{\partial w}{\partial \tau} - 6w\frac{\partial w}{\partial z} + \frac{\partial^3 w}{\partial z^3} = 0. \tag{1}$$

The KdV Eq. (1) was discovered in 1895 by Korteweg and de Vries [10]. This equation is a paradigmatic partial differential equation in nonlinear physics. It models one-dimensional shallow water waves with small but finite amplitudes. It has also been used to describe a number of important physical phenomena such as solitons, magnetohydrodynamics waves in warm plasma, acoustic waves in anharmonic crystals, ion-acoustic waves [33,36,1,17,3,24].

Eq. (1) is integrable and the Cauchy problem for this equation can be solved [7,14,9]. This equation has soliton, rational and elliptic solutions [2,27].

A suitable modification of the inverse scattering transform helps to shown that under the KdV flow an initial profile supported on $(-\infty, 0)$ instantaneously evolves into a meromorphic function with no poles on the real line [28].

Inverse scattering transform method is exploited for explicit computation of the shapes and the speeds of the asymptotic solitons for the KdV equation for several types of linearizable boundary conditions in [6].

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Many different methods for the construction of exact solutions of nonlinear evolution equations have been established and developed. The inverse scattering transform [1], the Hirota's bilinear operators [9], the Jacobi elliptic function expansion [16], the Darboux transformation [19], the Backlund transformation method [34] are successfully used to construct analytical solutions of nonlinear evolutions. A number of methods based on the extensive use of symbolic computations have been developed during the last decades. Homogeneous balance method [32], the Exp-function method [8,4], the tanh method [18,25] and its various extensions [5,15,31], the (G′/G) expansion method [38], the auxiliary (or subsidiary) ordinary differential equation method [35,30,37] are successfully used to solve high-dimensional nonlinear evolutions in mathematical physics with the help of symbolic computation. The key idea of most of these methods is that the traveling wave solution of a complicated nonlinear evolution equation can be guessed (supposed) as a polynomial (or a ratio of polynomials) of standard functions whose argument is a traveling wave term. The degree of the polynomial can be determined by considering homogeneous balance between the highest derivatives and nonlinear terms in the nonlinear evolution equation considered. Nevertheless, it can be noted that a straightforward application of these methods has attracted a considerable amount of criticism [12,23,11,22,13,26].

1.1. The purpose of the paper

The main objective of this paper is to show that well-known solitary solutions to KdV equation do not hold for all initial conditions. Formal operator techniques based on the generalized differential operator are used for the KdV equation to derive solutions expressed in a ratio of finite sums of standard functions [21]. We use an analytical criterion based on the concept of H-ranks and determine if a solution of the KdV equation can be expressed in an analytical form comprising standard functions. The employment of this criterion does not only give an answer to the above-stated question but gives the structure of the solution so that one does not have to guess what the form of the solution is. The load of symbolic calculations is brought before the structure of the solution is identified. This is in contrary to the methods described in [32,8,4,18,25,5,15,31,38,35,30,37] where the structure of the solution is first proposed, and then symbolic calculations are exploited for the identification of parameters.

We perform well-known transformations [29] before commencing the analysis of Eq. (1). The change of variables based on the existence of a propagating wave $w(z,\tau) = \omega(z - r\tau)$ (where r is the constant wave speed) yields an ordinary differential equation:

$$-(r + 6\omega)\omega'_x + \omega'''_{xxx} = 0, \tag{2}$$

where the prime mark stands for a derivative of ω with respect to the variable of the propagating wave $x = z - rt$. Integration of Eq. (2) with respect to x yields:

$$-(r + 3\omega)\omega + \omega''_{xx} = 3C; \quad C = \text{const.} \tag{3}$$

Eq. (3) which can be rewritten in the following form:

$$\omega''_{xx} = \gamma^2(\omega - \omega_1)(\omega - \omega_2); \quad \omega_1, \omega_2, \gamma \in \mathbf{C}, \tag{4}$$

where $\omega_{1,2} = -\frac{r}{6} \pm \sqrt{\frac{r^2}{36} - C}$; $\gamma^2 = 3$. We will seek structural solutions of Eq. (4) using algebraic operator techniques.

Elliptic function solutions (doubly periodic solutions) play an important role in studying exact solutions of nonlinear wave equations. Particularly, the Weierstrass elliptic function expansion method is used to seek doubly periodic solutions of the KdV equation, the Generalized KdV equation and the modified KdV equation in [39–42].

Note that Eq. (4) is the simple reduction of the KdV equation in the traveling waves. Eq. (4) can be integrated once more, and the resulting differential equation can be solved in terms of Weierstrass elliptic functions. Such a straightforward integration produces the solitary solution but does not produce conditions of its existence in the space of initial conditions [39–42]. The main goal of this paper is to demonstrate that algebraic operator techniques can simultaneously produce solitary solutions and conditions of their existence. And even though the KdV equation is one of the most studied nonlinear differential equations, we argue that conditions of existence of solitary solutions to the KdV equation cannot be produced by the Weierstrass elliptic function expansion method; formal application of algebraic operator techniques is necessary for that purpose.

1.2. Outline

The paper is organized as follows. Initial definitions and concepts used in the search of structural solutions are given in Section 2; the special solutions of KdV type ordinary differential equation are derived in Section 3; results of computational experiments are discussed in Section 4 and concluding remarks are given in Section 5.

2. Preliminaries

Several definitions and statements which will be exploited in the process of the search of special solutions of the KdV equation are concisely presented in this section.

2.1. Functions and their extensions

Functions of two types are used in this paper. Functions of the first type $p_j = p_j(c, s_0, \ldots, s_{n-1})$ describe the mapping $p_j : \mathrm{I}_\kappa \times \mathrm{I}_{s_0} \times \cdots \times \mathrm{I}_{s_{n-1}} \to R$; where $\mathrm{I}_\kappa, \mathrm{I}_{s_0}, \ldots, \mathrm{I}_{s_{n-1}} \subset R$ are variation intervals (or unions of intervals) of variables $c, s_0, \ldots, s_{n-1} \in R$. These functions are differentiable any number times in respect of every variable. It can be noted that the identification of variation intervals (or unions of intervals) is a straightforward task whenever the expression of $p_j(c, s_0, \ldots, s_{n-1})$ is given explicitly. For example, the function

$$p(c, s_0) = \frac{1}{c\left(1 + \sqrt{1 - 4s_0}\right)} \tag{5}$$

is defined and differentiable any number of times in respect of κ and s_0 when $c \in (-\infty; 0) \bigcup (0; +\infty)$ and $s_0 \in \left(-\infty; \frac{1}{4}\right)$ (the principal square root is considered in (5)). The analysis of functions of the first type is not the objective of this paper, but we will consider only such functions $p_j = p_j(c, s_0, \ldots, s_{n-1})$ that intervals $\mathrm{I}_\kappa, \mathrm{I}_{s_0}, \ldots, \mathrm{I}_{s_{n-1}}$ exist and are not empty sets. The set of functions of the first type is denoted as $\Phi_{\kappa, s_0, s_1, \ldots, s_{n-1}}$.

Functions of the second type are constructed from functions of the first type using the following algorithm.

(i) Construct the power series:

$$f_0(x, c, s_0, \ldots, s_{n-1}) = \sum_{j=0}^{+\infty} \frac{x^j}{j!} p_j(c, s_0, \ldots, s_{n-1}), \tag{6}$$

such that its domain of convergence $|x| < R \leqslant +\infty$ is nonempty and the convergence radius R depends on the properties of functions of the first type $\{p_j = p_j(c, s_0, \ldots, s_{n-1});\ j = 0, 1, 2, \ldots\}$.

(ii) Extend the function $f_0(x, c, s_0, \ldots, s_{n-1})$ to a wider domain (if it is possible) using classical extension techniques. The extended function $f(x, c, s_0, \ldots, s_{n-1})$ is denoted as the second type function.

For example, the series

$$f_0(x, c, s_0) = \sum_{j=0}^{+\infty} \frac{x^j}{j!} \left(j! \left(\frac{1}{c\left(1 + \sqrt{1 - 4s_0}\right)} \right)^j \right) = \sum_{j=0}^{+\infty} \left(\frac{x}{c\left(1 + \sqrt{1 - 4s_0}\right)} \right)^j$$

can be extended to a function

$$f(x, \kappa, s_0) = \frac{c\left(1 + \sqrt{1 - 4s_0}\right)}{c\left(1 + \sqrt{1 - 4s_0}\right) - x}$$

for $s_0 \in (-\infty; 0)$ and $c\left(1 + \sqrt{1 - 4s_0}\right) \neq x$. From now on we will use the equality

$$\sum_{j=0}^{+\infty} \frac{x^j}{j!} \left(j! \left(\frac{1}{c\left(1 + \sqrt{1 - 4s_0}\right)} \right)^j \right) = \frac{c\left(1 + \sqrt{1 - 4s_0}\right)}{c\left(1 + \sqrt{1 - 4s_0}\right) - x}$$

assuming that the transformation into the extended function does not cause any misunderstandings and will not specify domains for x, c and s_0.

Other forms of the second type functions can be used. Typical cases (structures of f_0) are listed below:

$$f_0(x, c, s_0, \ldots, s_{n-1}) = \sum_{j=0}^{+\infty} \frac{(x - c)^j}{j!} p_j(c, s_0, \ldots, s_{n-1});$$

$$f_0(c, s_0, \ldots, s_{n-1}) = \sum_{j=0}^{+\infty} \frac{c^j}{j!} p_j(c, s_0, \ldots, s_{n-1}).$$

It can be noted that it is not necessary to introduce the function norm (neither for the first type functions nor for the second type functions) in the process of the construction of analytic solutions of nonlinear ordinary differential equations.

The set of extended functions is denoted as $\Phi_{x, c, s_0, \ldots, s_{n-1}}(\Phi_{c, s_0, \ldots, s_{n-1}} \subset \Phi_{x, c, s_0, \ldots, s_{n-1}})$.

We will use several standard functions for the construction of generalized solutions of differential equations:

$$y_1(x) = \sum_{j=0}^{+\infty} \frac{x^j}{j!}; \quad y_1(x) = \exp(x);\ x \in R; \quad (y_1(x))_x^{(n)} = \exp(x). \tag{7}$$

$$y_2(x) = \sum_{j=0}^{+\infty} x^j;\ |x| < 1; \quad y_2(x) = \frac{1}{1 - x};\ x \neq 1; \quad (y_2(x))_x^{(n)} = \frac{n!}{(1 - x)^{1+n}}. \tag{8}$$

2.2. The H-rank of the sequence and the characteristic Hankel equation

Let $(p_j;\ j \in Z_0)$ is a sequence of numbers or functions. Then the corresponding sequence of Hankel matrixes $(H_1, H_2, \ldots)$ reads:

$$H_1 := [p_0]; \quad H_2 := \begin{bmatrix} p_0 & p_1 \\ p_1 & p_2 \end{bmatrix}; \quad H_3 := \begin{bmatrix} p_0 & p_1 & p_2 \\ p_1 & p_2 & p_3 \\ p_2 & p_3 & p_4 \end{bmatrix}; \quad \ldots \tag{9}$$

and the sequence of determinants of these matrixes is denoted as $(d_k; k \in N)$; where $d_k := \det H_k$.

Definition 1. A sequence $\left(p_j;\ j \in Z_0\right)$ has an *H*-rank equal to m if $d_m \neq 0$ but $d_{m+n} = 0$ for all $n \in N$. The following notion will be used throughout the manuscript:

$$Hr\left(p_j; j \in Z_0\right) = m. \tag{10}$$

Let Eq. (10) holds for a sequence $(p_j; j \in Z_0)$. Then it is possible to construct a characteristic *H*-equation [20]:

$$\det \begin{bmatrix} p_0 & p_1 & \cdots & p_m \\ p_1 & p_2 & \cdots & p_{m+1} \\ & & \cdots & \\ p_{m-1} & p_m & \cdots & p_{2m-1} \\ 1 & \rho & \cdots & \rho^m \end{bmatrix} = 0. \tag{11}$$

Definition 2. Roots $\rho_1, \rho_2, \ldots, \rho_m$ of the characteristic *H*-Eq. (11) are *H*-eigenvalues of the sequence $\left(p_j; j \in Z_0\right)$.

Theorem 1. *Let the characteristic H-Eq. (11) has n different H-eigenvalues $\rho_1, \rho_2, \ldots, \rho_n; (n \leqslant m)$; the recurrence indexes of ρ_k are $m_k;\ (m_k \in N);\ m_1 + m_2 + \cdots + m_n = m$. Then,*

$$p_j = \sum_{r=1}^{n} \sum_{k=0}^{m_r - 1} \mu_{rk} \binom{j}{k} \rho_r^{j-k}; \quad j = 0, 1, 2, \ldots, \tag{12}$$

where μ_{rk} are appropriate coefficients; $\binom{j}{k} := \begin{cases} \frac{j!}{r!(j-k)!} & \text{when } j \geqslant k; \\ 0 & \text{when } j < k; \end{cases}$ *here it is assumed that* $0^0 := 1;\ 0 \cdot (\pm\infty) := 0.$

The proof of Theorem 1 is given in [20]. It can be noted that coefficients μ_{rk} can be determined solving a system of linear algebraic equations which consists from m different equalities of Eq. (12) (*H*-eigenvalues $\rho_1, \rho_2, \ldots, \rho_n$ and their recurrence indexes $m_1, m_2, \ldots, m_n$ must be determined beforehand). The simplest system is produced when the first m equalities of Eq. (12) are selected (for $j = 0, 1, \ldots, m-1$). But the same results can be produced for $j = j_1, j_2, \ldots, j_m$ where $0 \leqslant j_1 < j_2 < \cdots < j_m < +\infty$. Moreover, this system of linear algebraic equations has a unique solution.

Definition 3. A sequence $\left(p_j;\ j \in Z_0\right)$ whose elements $p_j;\ j = 0, 1, 2, \ldots$ can be expressed in the form (12) is called an algebraic progression.

2.3. Structures of analytical solutions

Let two polynomials are defined as follows:

$$P_1(c, s) = \sum_{k,l \in Z_0} a_{kl} c^k s^l; \quad P_2(c, s, t) = \sum_{k,l,r \in Z_0} b_{klr} c^k s^l t^r; \tag{13}$$

where a_{kl} and b_{klr} are fixed real (or complex) numbers; c, s and t are real (or complex) variables. Then it is possible to construct two ordinary differential equations with initial conditions:

$$y'_x = P_1(x, y); \tag{14}$$

where $y = y(x, c, s);\ y(c, c, s) = s$; and

$$\omega''_{xx} = P_2\left(x, \omega, \omega'_x\right); \tag{15}$$

where $\omega = \omega(x, c, s, t)$; $\omega(c, c, s, t) = s;\ \omega'_x(x, c, s, t)\big|_{x=c} = t$.

Usual differentiation operations in respect of variables x, c, s and t are denoted by symbols D_x, D_c, D_s and D_t. Then it is possible to construct generalized differential operators D_y and D_ω in respect of variablesy and ω [22]:

$$D_y := D_c + P_1(c,s)D_s; \tag{16}$$

$$D_\omega := D_c + tD_s + P_2(c,s,t)D_t. \tag{17}$$

Definition 4. D_y is the generalized differential operator of the differential equation (14) and D_ω is the generalized differential operator of the differential equation (15).

Generalized differential operators D_y and D_ω can be exploited to construct analytical solutions $y(x,c,s)$ and $\omega(x,c,s,t)$ of differential equations (14) and (15) [22]:

$$y = \sum_{j=0}^{+\infty} \frac{(x-c)^j}{j!} D_y^j s; \tag{18}$$

$$\omega = \sum_{j=0}^{+\infty} \frac{(x-c)^j}{j!} D_\omega^j s, \tag{19}$$

which converge in some nonempty surrounding $|x-c| < \varepsilon$ in the complex plane. Furthermore, functions $y = y(x,c,s)$ and $\omega = \omega(x,c,s,t)$ can be extended into the whole complex plane with the exception of possible singular points.

Definition 5. Terms $D_y^j s$ and $D_\omega^j s$ are denoted as coefficients of solutions of differential equations (14) and (15).

2.4. Structures of analytical algebraic solutions

It is important for many engineering applications to obtain analytical – algebraic representations of solutions of ordinary differential equations in the following form:

$$y = \sum_{r=1}^{m} \mu_r f_r((x-c)\rho_r), \tag{20}$$

where m is a finite constant; $m \in N$; μ_r, f_r and ρ_r are ordinary functions. We will use the H-rank and associated H-eigenvalues for the construction of special analytical – algebraic solutions.

Given the differential equation (14), let us denote:

$$D_y^j s := p_j; \quad j = 0,1,2,\ldots \tag{21}$$

Then the following theorem holds true.

Theorem 2. *The following three following statements are equivalent:*

(i) $p_j = j!\sum_{r=1}^{m}\sum_{k=0}^{m_r-1}\mu_{rk}\binom{j}{k}\rho_r^{j-k}$; $j = 0,1,2,\ldots$ *where* $\rho_k,\ \mu_{rk} \in F_{c,s}$; $m, m_r = 1,2,\ldots$; *moreover* $\mu_{r,(m_r-1)} \neq 0$.

(ii) $Hr\left(\frac{p_j}{j!}; j \in Z_0\right) = \sum_{r=1}^{n} m_r$ *and the recurrence indexes of roots* $\rho_r \in F_{cs}$ *of the characteristic H-equation are* m_r; $r = 1,2,\ldots,n$.

(iii) *There exist functions* $\mu_{rk} \in F_{cs}$; $\rho_r \in F_{cs}$; $r = 1,2,\ldots,m$; $k = 0,1,2,\ldots,m_r-1$; $m, m_r \in N$ *which satisfy following conditions:*

(a) $\mu_{r(m_r-1)} \neq 0$; $r = 1,2,\ldots,m$; *moreover* $\sum_{r=1}^{m}\mu_{r0} = s$;

(b) $D_y\rho_r = \rho_r^2$;

(c) $D_y\mu_{rk} = \sigma_{k\,m_r}\cdot(2k-1)\rho_r\mu_{rk} + \sigma_{(k+1)m_r}\cdot k\mu_{r(k+1)}$,

where $\sigma_{kl} := \begin{cases} 1, & \text{when } k \leqslant l; \\ 0, & \text{when } k > l; \end{cases} k,l \in Z_0.$

The rigorous proof of the Theorem 2 is given in [21]. It can be noted that analogous theorems hold for the differential equation (15) and differential equations of the same type.

Corollary 1. *Let the statement (i) of the Theorem 2 holds. Then the solution (18) of the differential equation (14) can be expressed in the following form:*

$$y = \sum_{j=0}^{+\infty}\frac{(x-c)^j}{j!}j!\sum_{r=1}^{m}\sum_{k=0}^{m_r-1}\mu_{rk}\binom{j}{k}\rho_r^{j-k} = \sum_{r=1}^{m}\sum_{k=0}^{m_r-1}\mu_{rk}\frac{(x-c)^k}{k!}\sum_{j=0}^{+\infty}\frac{(j+k)!}{j!}(\rho_r(x-c))^j = \sum_{r=1}^{m}\sum_{k=0}^{m_r-1}\frac{\mu_{rk}(x-c)^k}{(1-\rho_r(x-c))^{k+1}}, \tag{22}$$

because $(1-z)^{-(k+1)} = \sum_{j=0}^{+\infty}\binom{j+k}{j}z^j$ *when* $|z| < 1$.

2.5. Expanding and narrowing an ordinary differential equation

Let two differential equations are given:

$$y'_x = P_1(x, y); \tag{23}$$

where $y = y(x; c, s);\ y(c; c, s) = s$ with the generalized operator of differentiation:

$$D_y := D_c + P_1(c, s)D_s \tag{24}$$

and

$$\omega''_{xx} = P_2(x, \omega); \tag{25}$$

where $\omega = \omega(x; c, s, t)$; $\omega(c; c, s, t) = s;\ \omega'_x(x; c, s, t)\big|_{x=c} = t$ with the generalized operator of differentiation:

$$D_\omega := D_c + tD_s + P_2(c, s)D_t. \tag{26}$$

Theorem 3. *If the equality*

$$\frac{\partial P_1(c, s)}{\partial c} + P_1(c, s)\frac{\partial P_1(c, s)}{\partial s} = P_2(c, s) \tag{27}$$

holds, then

$$D^j_y s = D^j_\omega s\Big|_{t=P_1(v,s)}; \quad j = 0, 1, 2, \ldots \tag{28}$$

and, moreover, the following equality holds true:

$$\omega(x; c, s, P_1(c, s)) = y(x; c, s). \tag{29}$$

Rigorous proof of Theorem 2 is given in [23].

Definition 6. If Eq. (27) holds, then the differential equation (23) is the narrowed differential equation of (25), and the differential equation (25) is the expanded differential equation of (23).

2.6. Changing the independent variable of a differential equation

Let the following differential equation is given:

$$y'_x = P_1(y), \tag{30}$$

where $y := y(x) = y(x; c, s)$; $y(c; c, s) = s$. The independent variable x can be changed using the substitution [22]:

$$z := \exp(\alpha x); \quad x = \frac{1}{\alpha}\ln z. \tag{31}$$

Then,

$$y = y(x) = y\left(\frac{1}{\alpha}\ln z\right) := \hat{y}(z) = \hat{y}(\exp(\alpha x)) := \hat{y} \tag{32}$$

and, furthermore,

$$y'_x = \alpha z\hat{y}'_z; \quad \hat{c} := \exp(\alpha c). \tag{33}$$

Thus, the differential equation (30) is transformed to the following differential equation:

$$\hat{y}'_z = \frac{1}{\alpha z}P_1(\hat{y}); \tag{34}$$

where $\hat{y} = \hat{y}(z; \hat{c}, s)$; $\hat{y}(\hat{c}; \hat{c}, s) = s$. The relationship between the solutions of differential equations (30) and (34) reads:

$$y(x; c, s) = \hat{y}(\exp(\alpha x); \exp(\alpha c), s). \tag{35}$$

Definition 7. The differential equation (34) is the image differential equation of the differential equation (30).

2.7. The substitution of dependent variable for linear differential equation

Let the differential equation (14) with corresponding initial conditions be given. Then the function $\hat{\omega} = \hat{\omega}(x, c, \hat{s}, \hat{t})$ is described using following expressions

$$\omega := a\hat{\omega} + b; \quad \hat{\omega} = \frac{1}{a}(\omega - b); \tag{36}$$

where $a, b \in C$ $(a \neq 0)$ is the linear substitution from dependent variable ω to dependent variable $\hat{\omega}$.

Since expressions $\omega'_x = a\hat{\omega}'_x$; $a\hat{\omega}'_x = \frac{1}{a}\omega'_x$; $\omega''_{xx} = a\hat{\omega}''_{xx}$; $\hat{\omega}''_{xx} = \frac{1}{a}\omega''_{xx}$ hold true, the differential equation $\omega''_x = \widehat{P}_2(x, \hat{\omega}, \hat{\omega}'_x)$ with initial conditions $\hat{\omega}(c, c, \hat{s}, \hat{t}) = \hat{s}$ and $\hat{\omega}'_x(c, c, \hat{s}, \hat{t})\big|_{x=c} = \hat{t}$ $(\widehat{P}_2(c, \hat{s}, \hat{t}) = \frac{1}{a}P_2(c, a\hat{s} + b, a\hat{t}))$ is obtained. Relationships between initial conditions read $s = a\hat{s} + b$; $t = a\hat{t}$ or $\hat{s} = \frac{1}{a}(s - b)$; $\hat{t} = \frac{1}{a}t$.

Moreover,

$$\hat{\omega}(x, c, \hat{s}, \hat{t}) = \frac{1}{a}\left(\omega(x, c, a\hat{s} + b, a\hat{t}) - b\right),$$

or

$$\omega(x, c, \hat{s}, \hat{t}) = a\hat{\omega}\left(x, c, \frac{1}{a}(s - b), \frac{1}{a}t\right) + b.$$

Note that $D_{\hat{\omega}} = D_c + tD_{\hat{s}} + P_2(c, \hat{s}, \hat{t})D_{\hat{t}}$ and $\hat{\omega} = \sum_{j=0}^{+\infty} \frac{(x-c)^j}{j!} \widehat{D}^j_{\hat{\omega}} \hat{s}$.

3. Solution of the KdV type ordinary differential equation by the generalized operator method

3.1. The narrowed KdV equation

As noted in the Introduction, we will consider the following KdV type ordinary differential equation:

$$\omega''_{xx} = \gamma^2(\omega - \omega_1)(\omega - \omega_2) \tag{37}$$

with $\omega = \omega(x, c, s, t)$ and $\omega(c, c, s, t) = s$; $\omega'_x(x, c, s, t)\big|_{x=c} = t$; where γ, $\omega_1, \omega_2 \in \mathbf{C}$; $\gamma \neq 0$ are parameters of the differential equation.

Linear substitution (36) transforms differential equation (37) to the following differential equation

$$a\hat{\omega}''_{xx}, = \gamma^2(a\hat{\omega} + b - \omega_1)(a\hat{\omega} + b - \omega_2).$$

That is equivalent to

$$\hat{\omega}''_{xx}, = \hat{\gamma}^2(\hat{\omega} - \hat{\omega}_1)(\hat{\omega} - \hat{\omega}_2). \tag{38}$$

Eq. (38) is the KdV type differential equation where parameters read:

$$\hat{\gamma}^2 = \gamma^2 a; \quad \hat{\omega}_1 = \frac{\omega_1 - b}{a}; \quad \hat{\omega}_2 = \frac{\omega_2 - b}{a}. \tag{39}$$

Eq. (37) helps to find solutions of KdV type differential equations when parameters of this equation are associated by relationship (39).

The narrowed differential equation of Eq. (37) reads [23]:

$$y'_x = P_1(x, y); \quad y = y(x, c, s); \quad y(c, c, s) = s; \tag{40}$$

the expanded differential equation of Eq. (40) is Eq. (37) [23]. It follows from Eq. (37) that $P_2(c, s) = \gamma^2(s - \omega_1)(s - \omega_2) = P_2(s)$. Thus it is possible to construct the simplest narrowed equation by assuming $P_1(c, s) = P_1(s)$. The equality (27) yields the relationship $P_1(s) \cdot (P_1(s))'_s = P_2(s)$; the first integral of the latter produces:

$$(P_1(s))^2 = 2\gamma^2\left(\int (s - \omega_1)(s - \omega_2)ds - C_0\right), \tag{41}$$

where $C_0 \in$ Cis a fixed constant. Thus, the narrowed KdV equation takes the form:

$$(y'_x)^2 = \frac{2}{3}\gamma^2(y - y_1)(y - y_2)(y - y_3), \tag{42}$$

where

$$(s - y_1)(s - y_2)(s - y_3) = s^3 - \frac{3(\omega_1 + \omega_2)}{2}s^2 + 3\omega_1\omega_2 s - 3C_0. \tag{43}$$

Thus, the narrowed differential equation (42) can be expanded to KdV Eq. (32) for every C_0; $C_0 \in \mathbf{C}$. Moreover, the general solution of the differential (42) $y = y(x, c, s)$ is also a solution of KdV Eq. (37).

Note that it is possible to obtain Eq. (42) from (37) via multiplying (37) by ω'_x and integrating ones without any formulas for P_1 and P_2. Moreover, Eq. (42) can be reduced to equation which is solvable in terms of Weierstrass elliptic functions. But the presented transformations are necessary to illustrate the functionality of the proposed technique which helps to derive explicit conditions of the existence of solitary solutions in the space of initial conditions and system parameters.

3.2. Conditions when the solution of KdV equation can be expressed in a ratio of sums of exponential functions

We will find such values of the constant C_0 with which solutions of the differential equation (42) $y = y(x, c, s)$ can be expressed in a ratio of sums of exponential functions. The image differential equation of (42) is constructed for that purpose [22]:

$$\left(\alpha z \hat{y}'_z\right)^2 = \frac{2}{3}\gamma^2(\hat{y} - y_1)(\hat{y} - y_2)(\hat{y} - y_3), \tag{44}$$

where $\hat{y} = \hat{y}(z, \hat{c}, s)$; $\hat{y}(\hat{c}, \hat{c}, s) = s$ and $z := e^{\alpha x}$; $\hat{c} = e^{\alpha c}$. Then, the generalized differential operator [22,21] reads:

$$D_{\hat{y}} := D_{\hat{c}} + \sqrt{\frac{2}{3}}\frac{\gamma}{\alpha\hat{c}}\sqrt{(s - y_1)(s - y_2)(s - y_3)} \cdot D_s. \tag{45}$$

Let $\hat{p}_j := D^j_{\hat{y}} s$; $j = 0, 1, 2, \ldots$. The sequence of Hankel matrixes [22] takes the following form:

$$H_m := \begin{bmatrix} \frac{\hat{p}_0}{0!} & \frac{\hat{p}_1}{1!} & \cdots & \frac{\hat{p}_{m-1}}{(m-1)!} \\ \frac{\hat{p}_1}{1!} & \frac{p_2}{2!} & \cdots & \frac{\hat{p}_m}{m!} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\hat{p}_{m-1}}{(m-1)!} & \frac{\hat{p}_m}{m!} & \cdots & \frac{\hat{p}_{2m-2}}{(2m-2)!} \end{bmatrix}; \quad m = 1, 2, 3, \ldots \tag{46}$$

Symbolic computations help to prove that $\det H_4 = 0$ when $y_1 = y_2 := y_0$; $y_0 \neq y_3$ and the following equality holds (Appendix A):

$$3\alpha^2 = 2\gamma^2(y_0 - y_3). \tag{47}$$

The differential equation (44) takes the following form when the equality (47) holds:

$$(y_0 - y_3)\left(z\hat{y}'_z\right)^2 = (\hat{y} - y_0)^2(\hat{y} - y_3) \tag{48}$$

and its narrowed differential equation reads:

$$3\left(y'_x\right)^2 = 2\gamma^2(y - y_0)^2(y - y_3). \tag{49}$$

Then, relationships in Eqs. (43) and (49) yield:

$$\begin{cases} 2y_0 + y_3 = \frac{3(\omega_1+\omega_2)}{2} \\ y_0^2 + 2y_0y_3 = 3\omega_1\omega_2 \\ y_0^2 y_3 = 3C_0. \end{cases} \tag{50}$$

Eq. (50) produces following relationships between y_0, y_3 and ω_1, ω_2:

$$y_0 = \omega_1; \quad y_3 = \frac{3\omega_2 - \omega_1}{2}; \quad c = \frac{3\omega_2\omega_1^2 - \omega_1^3}{3};$$
$$y_0 - y_3 = \frac{3}{2}(\omega_1 - \omega_2); \quad s - y_3 = \frac{1}{2}(2s - 3\omega_2 + \omega_1). \tag{51}$$

Thus two narrowed KdV differential equations are produced:

$$3\left(y'_x\right)^2 = 2\gamma^2(y - \omega_i)^2\left(y - \frac{3\omega_j - \omega_i}{2}\right); \quad i, j = 1, 2; \; i \neq j, \tag{52}$$

whose solutions could be expressed in a ratio of sums of exponential functions (what is the objective of the next section).

3.3. The solution of the narrowed differential equation

The generalized operator of differentiation of the narrowed differential equation of (48) reads:

$$D_{\hat{y}} = D_{\hat{c}} + \frac{s - y_0}{\hat{c}}\sqrt{\frac{s - y_3}{y_0 - y_3}}D_s. \tag{53}$$

Then, the Hankel characteristic equation [22] takes the form:

$$\det\left.\begin{bmatrix} \frac{\hat p_0}{0!} & \frac{\hat p_1}{1!} & \frac{\hat p_2}{2!} & \frac{\hat p_3}{3!} \\ \frac{\hat p_1}{1!} & \frac{\hat p_2}{2!} & \frac{\hat p_3}{3!} & \frac{\hat p_4}{4!} \\ \frac{\hat p_2}{2!} & \frac{\hat p_3}{3!} & \frac{\hat p_4}{4!} & \frac{\hat p_5}{5!} \\ 1 & \hat\rho & \hat\rho^2 & \hat\rho^3 \end{bmatrix}\right|_{\substack{y_1=y_2:=y_0 \\ 3\alpha^2-2\gamma^2(y_0-y_3)=0}} = 0 \tag{54}$$

and its roots are (Appendix B):

$$\hat\rho_1 = 0; \quad \hat\rho_2 = \hat\rho_3 = \frac{1}{2\hat c}\left(\sqrt{\frac{s-y_3}{y_0-y_3}} - 1\right). \tag{55}$$

It can be noted, that $D_{\hat y}\hat\rho_j = \hat\rho_j^2;\ j = 1,2,3$ (Appendix C).

The linear algebraic system for the determination of $\mu_j;\ j = 1,2,3$ takes the following form when $y_0 = y_1 = y_2$ and Eq. (47) holds true [22]:

$$\begin{cases} \mu_1 + \mu_2 = \hat p_0 \\ \hat\rho_2\mu_2 + \mu_3 = \hat p_1 \\ \hat\rho_2^2\mu_2 + 2\hat\rho_2\mu_3 = \frac{1}{2}\hat p_2 \end{cases} \tag{56}$$

and its solution reads:

$$\mu_1 = y_0; \quad \mu_2 = s - y_0; \quad \mu_3 = \frac{s-y_0}{2\hat c}\left(\sqrt{\frac{s-y_3}{y_0-y_3}} + 1\right). \tag{57}$$

We will prove that expressions of $\mu_j;\ j = 1,2,3$ satisfy following equalities (Appendix D):

$$D_{\hat y}\mu_1 = \rho_1\mu_1; \quad D_{\hat y}\mu_2 = \rho_2\mu_2 + \mu_3; \quad D_{\hat y}\mu_3 = 3\rho_2\mu_3. \tag{58}$$

Thus,

$$\frac{1}{j!}D_{\hat y}^j s = \mu_1\hat\rho_1^j + \mu_2\hat\rho_2^j + \mu_3 j\hat\rho_2^{j-1}; \tag{59}$$

the proof is given in (Appendix E). Then, according to Theorem 3, Corollary 1 and Eq. (59), the solution of the differential equation (48) reads:

$$\hat y(z;\hat c,s) = y_0 + \frac{4(s-y_0)(y_0-y_3)z\hat c}{\left(\left(\sqrt{s-y_3}+\sqrt{y_0-y_3}\right)\hat c - \left(\sqrt{s-y_3}-\sqrt{y_0-y_3}\right)z\right)^2}; \tag{60}$$

the detailed derivation is given in (Appendix F). Finally, the structure of the solution of the narrowed differential equation (49) reads:

$$y(x;c,s) = y_0 + \frac{4(s-y_0)(y_0-y_3)\exp\left(\sqrt{\frac{2}{3}}\gamma\sqrt{y_0-y_3}(x-c)\right)}{\left(\left(\sqrt{s-y_3}+\sqrt{y_0-y_3}\right) - \left(\sqrt{s-y_3}-\sqrt{y_0-y_3}\right)\exp\left(\sqrt{\frac{2}{3}}\gamma\sqrt{y_0-y_3}(x-c)\right)\right)^2}. \tag{61}$$

It can be noted that Eq. (61) represents the general structure of the solution only. Concrete special solutions are enumerated at the beginning of the next section when all possible combinations of signs of corresponding square roots are placed accordingly.

3.4. Solutions of the KdV type ordinary differential equation and conditions of their existence

Relationships (51) yield special solutions $y_{ij},\ i,j = 1,2,\ i \neq j$, of the narrowed KdV Eq. (52) and (37):

$$\begin{aligned} &\omega\left(x;c,s,\left(\pm\frac{1}{\sqrt3}\gamma(s-\omega_r)\sqrt{2s+\omega_r-3\omega_j}\right)\right) = y_{rj}(x;c,s) \\ &= \omega_r + \frac{12(\omega_r-\omega_j)(s-\omega_r)\exp\left(\gamma\sqrt{\omega_r-\omega_j}(x-c)\right)}{\left(\left(\sqrt{3(\omega_r-\omega_j)}\pm\sqrt{2s-3\omega_j+\omega_r}\right)+\left(\sqrt{3(\omega_r-\omega_j)}\mp\sqrt{2s-3\omega_j+\omega_r}\right)\exp\left(\gamma\sqrt{\omega_r-\omega_j}(x-c)\right)\right)^2};\ r,j=1,2;\ r\neq j; \end{aligned} \tag{62}$$

this solution exists on the curve

$$3t_{rj}^2 = \gamma^2(s-\omega_r)^2(2s-3\omega_j+\omega_r); \quad r,j = 1,2;\ r \neq j. \tag{63}$$

And though it can be observed that the obtained solution (62) has the well-known form

$1/(c_1 \sinh ax + c_2 \cosh ax)^2 + c_3,$

the explicit expression of the curve (63) is a new and important finding describing the existence of solitary solutions in the space of initial conditions and system parameters.

As mentioned in the Introduction, explicit expressions of analytic solutions of KdV equation can be found using different methods and techniques based on the proposition that the structure of the solution takes certain form and then using symbolic computations to identify the necessary values [32,8,4,18,25,5,15,31,38,35,30,37]. But we argue that conditions of the existence of these special solutions in the space of the system's parameters can not be derived using those methods.

Note that another differential equations of KdV type can be obtained using substitutions (35) and relationships (39) for parameters $\gamma,\ \omega_1,\ \omega_2$ and $\hat{\gamma},\ \hat{\omega}_1,\ \hat{\omega}_2$.

3.5. Several comments on special solutions of the KdV type ordinary differential equation

Special solutions of the KdV Eq. (62) exist and have sense not only when parameters $\gamma,\ \omega_1$ and ω_2 are complex numbers, but when the parameter of the initial condition s is a complex number also. Thus, in general, solutions (Eq. 62) are complex functions of the real variable x:

$$y_k = u_k(x; c, s) + i v_k(x; c, s); \tag{64}$$

where $k = 1, 2;\ i^2 = -1$. Then Eq. (37) yields a system of KdV differential equations whose special solutions are functions $u_k = u_k(x; c, s)$ and $v_k = v_k(x; c, s);\ k = 1, 2$. But a more deep analysis of solutions u_k and v_k is out of scope of interests in this paper. In general, the independent variable x can be a complex variable then; complex variable functions theory methods could be exploited for the analysis of special complex solutions.

In this paper we will introduce the limitation that $\gamma, \omega_1,\ \omega_2$ and s are real numbers. It can be noted that functions y_k in Eq. (64) are complex functions even if the aforementioned limitation is in force; special solutions satisfy the following system of KdV differential equations:

$$\begin{cases} (u_k)''_{xx} = \gamma^2\left((u_k - \omega_1)(u_k - \omega_2) - v_k^2\right); \\ (v_k)''_{xx} = \gamma^2(2u_k - (\omega_1 + \omega_2))v_k; \end{cases} \tag{65}$$

where $k = 1, 2$ and functions u_k, v_k satisfy initial conditions:

$$u_k(v; v, s) = s; \quad v_k(v; v, s) = 0. \tag{66}$$

Functions u_k and v_k are structural solutions of the system of differential equation (65). Therefore, their first derivatives must fulfill the following equalities defined by Eq. (63):

$$\left.\frac{d(u_k(x; v, s))}{dx}\right|_{x=c} = a_k; \quad \left.\frac{d(v_k(x; v, s))}{dx}\right|_{x=c} = b_k, \tag{67}$$

where $t_k = a_k + ib_k;\ k = 1, 2;\ a_k$ and b_k are real numbers computed from the equations of curves defined by Eq. (63).

4. Classical KdV solitary solutions and numerical experiments

Let us assume that $\omega_1 = 0$ and $\omega_2 = \frac{2}{3}s$. Then, Eq. (62) yields:

$$y_{11}(x; c, s) = \frac{4s \exp\left(\gamma\sqrt{-\omega_2}(x - c)\right)}{\left(1 + \exp\left(\gamma\sqrt{-\omega_2}(x - c)\right)\right)^2} = s \cdot \operatorname{sech}^2\left(\frac{\gamma}{2}\sqrt{-\omega_2}(x - c)\right). \tag{68}$$

But if $\omega_1 = 0$ the second root must be equal to $\omega_2 = -\frac{r}{3}$ (Eq. (4)); also $\gamma^2 = 3$. Then,

$$y_{11} = -\frac{r}{2} \cdot \operatorname{sech}^2\left(\frac{\sqrt{r}}{2}(x - c)\right) = -\frac{r}{2} \cdot \operatorname{sech}^2\left(\frac{\sqrt{r}}{2}(z - r\tau - c)\right), \tag{69}$$

what is a solution of the KdV equation describing solitary waves observed by Rassel in 1834.

It is important to note, that solutions for other values on ω_1 and ω_2 may not exist in the whole plane of parameters s and t. We argue that these conditions of existence can not be determined using methods based on the principle when the structure of the solution is initially supposed and then symbolic computations are used for the determination of appropriate coefficients. We have shortly discussed the limitations of these methods in the Introduction. We will now demonstrate that solutions for the other set of parameters ω_1 and ω_2 are limited by their conditions of existence.

Let us select $\omega_1 = 0.5,\ \omega_2 = 3$ and $\gamma^2 = 3$ for example. Conditions of existence of the solution y_{21} (Eqs. (62) and (63)) are shown in Figs. 1 and 2. Next, we integrate the differential equation $\omega''_{xx} = 3(\omega - 0.5)(\omega - 3)$ using numerical time marching techniques assuming different initial conditions s and t. At the same time we calculate numerical values of the solution in Eq. (62) and compute cumulative errors between the numerical and analytical solutions (the detailed procedure for the calculation of errors is described in [23]). We limit the analysis to the real solutions only, so we visualize errors in the region

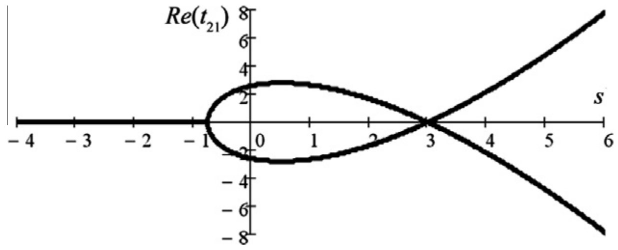

Fig. 1. Conditions of existence of the solution y_{21} in the parameter plane of initial conditions; $\omega_1 = 0.5,\ \omega_2 = 3;\ \gamma^2 = 3$ (real parts of t_{21} are represented as solid lines).

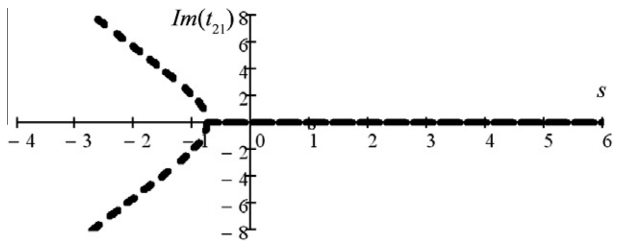

Fig. 2. Conditions of existence of the solution y_{21} in the parameter plane of initial conditions; $\omega_1 = 0.5,\ \omega_2 = 3;\ \gamma^2 = 3$ (imaginary parts of t_{21} are represented as dashed lines).

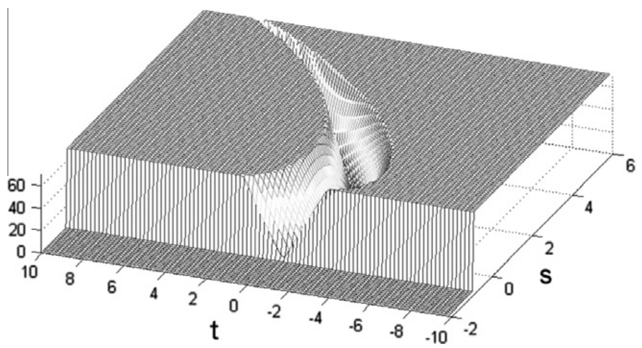

Fig. 3. The surface of errors computed as differences between 100 steps of the numerical solution of the differential equation $\omega''_{xx} = 3(\omega - 0.5)(\omega - 3)$ and the analytic solution y_{21} at $\omega_1 = 0.5,\ \omega_2 = 3;\ \gamma^2 = 3$ in the parameter plane of initial conditions; errors greater that 70 are truncated to 70 for clarity.

$s \geqslant -0.75$ only. The produced trench in the surface of cumulative errors (Fig. 3) accurately follows one of the branches of the solution existence graph in Fig. 1.

5. Concluding remarks

The proposed algebraic operator method for the construction of solitary solutions has a number of important and advantageous features:

(i) This method allows determining the existence of solitary solutions (a solitary solution does exist if the H-rank of the solution to the image differential equation does exist).
(ii) This method does not require to guess (or to suppose) the structure of the solitary solution (the structure of the solitary solution is generated automatically).
(iii) This method does generate not only the structure of the solitary solution. Explicit conditions of the existence of these solutions in the subspace of systems parameters and initial conditions are also generated automatically.

We used the KdV equation as a typical differential equation for the demonstration of the functionality of the proposed approach. It is true that the KdV equation is one of the most studied nonlinear differential equations. But such a choice helps to reveal the power of the proposed method since it is possible to make straightforward comparisons with existing techniques.

We show that special solitary solutions of the KdV equation exist only on a line in the parameter plane of initial conditions – if only that solution does not coincide with the solution expressible in the sech function. This is a new theoretical result which may lead to important findings in a variety of practical applications where the generation or manipulation with solitary waves could be considered.

Acknowledgment

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Appendix A

The notion

$$\frac{1}{\beta} := \sqrt{\frac{2}{3}\frac{\gamma}{\alpha}} \tag{A.1}$$

is introduced before computing the expression of $\det H_4$. Then, the generalized operator of differentiation defined in Eq. (45) takes the form:

$$D_{\hat{y}} = D_{\hat{c}} + \frac{\sqrt{(s-y_1)(s-y_2)(s-y_3)}}{\beta\hat{c}} D_s \tag{A.2}$$

and coefficients $p_0, p_1, \ldots, p_6$ read:

$$p_0 = s; \quad p_1 = \frac{\sqrt{(s-y_1)(s-y_2)(s-y_3)}}{\beta\hat{c}}; \ldots \tag{A.3}$$

Symbolic computations are exploited to determine coefficients $p_0, p_1, \ldots, p_6$; we omit expressions of higher coefficients for the brevity. Unfortunately, it turns out that the expression of higher coefficients is lengthy and ordinary symbolic computation systems can not derive the explicit expression of $\det H_4$ when common personal computers are used. Thus, we form a subsequence $(p_1, p_2, \ldots, p_5)$ and compute $\det \widetilde{H}_3$ instead, where

$$\widetilde{H}_3 = \begin{bmatrix} \frac{p_1}{1!} & \frac{p_2}{2!} & \frac{p_3}{3!} \\ \frac{p_2}{2!} & \frac{p_3}{3!} & \frac{p_4}{4!} \\ \frac{p_3}{3!} & \frac{p_4}{4!} & \frac{p_5}{5!} \end{bmatrix}. \tag{A.4}$$

Now symbolic computations yield:

$$\det \widetilde{H}_3 = \frac{\sqrt{(s-y_1)(s-y_2)(s-y_3)}}{2160\beta^9\hat{c}^9} \sum_{k=0}^{4} A_k(y_1, y_2, y_3, \beta)s^k; \tag{A.5}$$

where

$$A_4(y_1,y_2,y_3,\beta) = \frac{27}{16}\left(y_1^2 + y_2^2 + y_3^2 - y_1y_2 - y_1y_3 - y_2y_3 - \beta^4\right);$$

$$\begin{aligned} A_3(y_1,y_2,y_3,\beta) = &\frac{1}{4}y_1^3 + \frac{1}{4}\left(6\beta^2 - 15(y_2+y_3)\right)y_1^2 + \frac{1}{4}\left(-15y_2^2 + (87y_3 - 6\beta^2)y_2 + 9(\beta^2+y_3)\left(\beta^2 - \frac{5}{3}y_3\right)\right)y_1 \\ &+ \frac{1}{4}\left(y_2 + y_3 + \beta^2\right)\left(y_2^2 + (5\beta^2 - 16y_3)y_2 + (\beta^2+y_3)(y_3 + 4\beta^2)\right); \end{aligned}$$

$$\begin{aligned} A_2(y_1,y_2,y_3,\beta) = &-\frac{1}{4}y_1^4 + \frac{1}{8}\left(y_2 + y_3 - 4\beta^2\right)y_1^3 + \frac{1}{8}\left(87y_2^2 - 87y_2y_3 + 87y_3^2 - 18\beta^4\right)y_1^2 \\ &+ \frac{1}{8}\left(y_2^3 - 87y_2^2y_3 + (36y_3\beta^2 - 87y_3^2 - 9\beta^2)y_2 - 8\beta^6 - 9y_3\beta^4 + y_3^3\right)y_1 \\ &+ \frac{1}{8}\left(-2y_2^4 + (y_3 - 12\beta^2)y_2^3 + (87y_3^2 - 18\beta^4)y_2^2 + (y_3^3 - 9y_2\beta^4 - 8\beta^6)y_2 - (\beta^2+y_3)^2 y_3(2y_3 + 8\beta^2)\right); \end{aligned}$$

$$\begin{aligned} A_1(y_1,y_2,y_3,\beta) = &\frac{1}{4}(y_2+y_3)y_1^4 + \frac{1}{4}\left(-15y_2^2 + (25y_3 + 6\beta^2)y_2 + 3(2\beta^2 - 5y_3)y_3\right)y_1^3 \\ &+ \frac{1}{4}\left(-15y_2^3 + (3y_3 - 6\beta^2)y_2^2 + 3(y_3^2 - 2y_3\beta^2 + 3\beta^4)y_2 + 9(y_3+\beta^2)\left(\beta^2 - \frac{5}{3}y_3\right)y_3\right)y_1^2 \\ &+ \frac{1}{4}\left(y_2^4 + (6\beta^2 + 25y_3)y_2^3 + 3(y_3^2 - 2y_3\beta^2 + 3\beta^4)y_2^2 + (4\beta^4 - 31y_3\beta^2 + 25y_3^2)(\beta^2+y_3)y_2 + (\beta^2+y_3)^2(y_3 + 4\beta^2)y_3\right)y_1 \\ &+ \frac{1}{4}\left(y_2^2 + (5\beta^2 - 16y_3)y_2 + (y_3+\beta^2)(y_3+4\beta^2)\right)(y_2+y_3+\beta^2)y_2y_3; \end{aligned}$$

$$\begin{aligned}A_0(y_1,y_2,y_3,\beta)=&\frac{1}{16}(27y_3^2+27y_2^2-58y_2y_3)y_1^4-\frac{1}{16}(27y_2^3-33y_2^2y_3+3y_2y_3(8\beta^2-11y_3)+27y_3^3)y_1^3\\&+\frac{1}{16}(27y_2^4+33y_2^3y_3+(24y_3\beta^2-27\beta^4-105y_3^2)y_2^2+(24\beta^2y_3^2+33y_3^3+18y_3\beta^4)y_2+27y_3^4-27\beta^4y_3^2)y_1^2\\&-\frac{1}{8}\left(29y_2^3+\left(12\beta^2-\frac{33}{2}y_3\right)y_2^2-(12y_3\beta^2+33y_3^2+9\beta^4)y_2+(\beta^2+y_3)(8\beta^4-17y_3\beta^2+29y_3^2)\right)y_1y_2y_3\\&-\frac{27}{16}(y_2y_3-y_2^2-y_3^2+\beta^4)y_2^2y_3^2.\end{aligned}$$

The necessary and sufficient conditions for $\det\widetilde{H}_3=0$ are equalities

$$A_k(y_1,y_2,y_3,\beta)=0;\quad k=0,1,2,3,4. \tag{A.6}$$

The equality

$$A_4(y_1,y_2,y_3,\beta)=0, \tag{A.7}$$

yields the following relationship:

$$\beta^4=y_1^2+y_2^2+y_3^2-y_1y_2-y_1y_3-y_2y_3. \tag{A.8}$$

Next, the equality (A.8) is incorporated into the expression of $A_3(y_1,y_2,y_3,\beta)$:

$$A_3\left(y_1,y_2,y_3,\sqrt[4]{y_1^2+y_2^2+y_3^2-y_1y_2-y_1y_3-y_2y_3}\right)=0. \tag{A.9}$$

Elementary transformations in Eq. (A.9) yield necessary and sufficient conditions for Eq. (A.8) ($\beta\neq0$) to hold. These conditions require that two of the three parameters y_1,y_2,y_3 must coincide, for example, $y_1=y_2:=y_0$ and $y_0\neq y_3$ (then $\beta=\sqrt{y_0-y_3}$). Now, the expressions of y_0 and β can be introduced into $A_k(y_1,y_2,y_3,\beta);\ k=0,1,2$. Elementary transformations help to prove that equalities in (A.6) hold true for all k.

Thus, $\det\widetilde{H}_3=0$ when $y_1=y_2:=y_0$, (A.1) and the equality $\beta=\sqrt{y_0-y_3}$ yield:

$$3\alpha^2=2\gamma^2(y_0-y_3). \tag{A.10}$$

Now, straightforward symbolic computations help to prove that $\det H_4=0$, but $\det H_3\neq0$.

Appendix B

Coefficients of the equality (54) read:

$$\hat{p}_0=s;$$

$$\hat{p}_1=\frac{s-y_0}{\hat{c}}\sqrt{\frac{s-y_3}{y_0-y_3}};$$

$$\hat{p}_2=\frac{3}{2}\frac{s-y_0}{\hat{c}^2(y_0-y_3)}\left(\sqrt{\frac{s-y_3}{y_0-y_3}}\left(-\frac{3}{2}y_0+\frac{2}{3}y_3\right)-\frac{1}{3}y_0-\frac{2}{3}y_3+s\right);$$

$$\hat{p}_3=3\frac{s-y_0}{\hat{c}^3(y_0-y_3)}\left(\sqrt{\frac{s-y_3}{y_0-y_3}}(s-y_3)+\frac{1}{2}y_0-\frac{3}{2}s+y_3-3s\right);$$

$$\hat{p}_4=\frac{15}{2}\frac{s-y_0}{\hat{c}^4(y_0-y_3)^2}\left(\sqrt{\frac{s-y_3}{y_0-y_3}}\left(-\frac{12}{5}\left(-\frac{1}{3}y_0-\frac{2}{3}y_3+s\right)(y_0-y_3)\right)-\frac{3}{5}y_0^2-\frac{16}{5}y_3s+\frac{8}{5}y_3^2+\frac{6}{5}y_0s+s^2\right);$$

$$\hat{p}_5=\frac{45}{2}\frac{s-y_0}{\hat{c}^5(y_0-y_3)^2}\left(\sqrt{\frac{s-y_3}{y_0-y_3}}\left(-\frac{5}{3}y_0^2+\frac{10}{3}y_0s+s^2+\frac{8}{3}y_3^2-\frac{16}{3}y_3s\right)+y_3\left(\frac{20}{3}s-\frac{4}{3}y_0\right)+\frac{2}{3}y_0^2-\frac{10}{3}s^2\right);$$

$$\begin{aligned}\hat{p}_6=&\frac{315}{4}\frac{s-y_0}{\hat{c}^6(y_0-y_3)^3}\left(\sqrt{\frac{s-y_3}{y_0-y_3}}\left(-\frac{30}{7}\left(-\frac{16}{15}y_3^2+y_3\left(\frac{8}{15}y_0-\frac{8}{3}s\right)-\frac{3}{5}y_0^2+\frac{2}{3}y_0s+s^2\right)(y_0-y_3)\right)+\frac{97}{7}y_3^2s-\frac{32}{7}y_3^3\right.\\&\left.+y_3\left(\frac{30}{7}y_0^2-\frac{60}{7}y_0s-\frac{66}{7}s^2\right)-\frac{5}{7}y_0^3-\frac{15}{7}y_0^2s+\frac{45}{7}y_0s^2+s^3\right)\end{aligned}$$

Eq. (54) then reads:

$$\frac{1}{64}\frac{y_0\hat{\rho}(s-y_0)^4}{\hat{c}^8(y_0-y_3)^4}\left(-2(\hat{\rho}\hat{c}+1)(y_0-y_3)\sqrt{\frac{s-y_3}{y_0-y_3}}\left(y_0\left(\frac{1}{2}+\hat{\rho}\hat{c}\right)-y_3-\hat{\rho}\hat{c}y_3+\frac{1}{2}s\right)+\frac{1}{4}s^2+y_0^2\left(\frac{1}{2}+\hat{\rho}\hat{c}\right)^2\right.$$
$$\left.+y_0\left(s\left(\hat{\rho}^2\hat{c}^2+3\hat{\rho}\hat{c}+\frac{3}{2}\right)-2y_3-5\hat{\rho}\hat{c}y_3-3\hat{\rho}^2\hat{c}^2y_3\right)-s(3\hat{\rho}\hat{c}y_3+3\hat{\rho}^2\hat{c}^2y_3+2y_3)+2y_3^2(\hat{\rho}\hat{c}+1)^2\right)=0.$$

Eq. (54) is solved using symbolic computation techniques; three roots read:

$$\hat{\rho}_1=0;\hat{\rho}_2=\hat{\rho}_3=-\frac{1}{2\hat{c}}\frac{-y_0+3y_0\sqrt{\frac{s-y_3}{y_0-y_3}}+4y_3-4y_3\sqrt{\frac{s-y_3}{y_0-y_3}}-3s+s\sqrt{\frac{s-y_3}{y_0-y_3}}}{-y_0+2y_0\sqrt{\frac{s-y_3}{y_0-y_3}}+2y_3-2y_3\sqrt{\frac{s-y_3}{y_0-y_3}}-s}=\frac{1}{2\hat{c}}\left(\sqrt{\frac{s-y_3}{y_0-y_3}}-1\right).$$

Appendix C

The equality $D_{\hat{y}}\hat{\rho}_1=\hat{\rho}_1^2$ is trivial because $\hat{\rho}_1=0$.

$\hat{\rho}_2=\hat{\rho}_3=\frac{1}{2\hat{c}}\left(\sqrt{\frac{s-y_3}{y_0-y_3}}-1\right)$, therefore:

$$D_{\hat{y}}\hat{\rho}_2=\left(D_{\hat{y}}+\frac{s-y_0}{\hat{c}}\sqrt{\frac{s-y_0}{y_0-y_3}}D_s\right)\frac{1}{2\hat{c}}\left(\sqrt{\frac{s-y_3}{y_0-y_3}}-1\right)=-\frac{1}{2\hat{c}^2}\left(\sqrt{\frac{s-y_3}{y_0-y_3}}-1\right)+\frac{s-y_0}{2\hat{c}^2}\sqrt{\frac{s-y_3}{y_0-y_3}}\frac{1}{2\sqrt{(y_0-y_3)(s-y_3)}}$$
$$=\frac{1}{4\hat{c}^2}\left(2+\frac{s-y_0}{y_0-y_3}-2\sqrt{\frac{s-y_3}{y_0-y_3}}\right)=\frac{1}{4\hat{c}^2}\left(\frac{s-y_3}{y_0-y_3}-2\sqrt{\frac{s-y_3}{y_0-y_3}}+1\right)=\left(\frac{1}{2\hat{c}}\left(\sqrt{\frac{s-y_0}{y_0-y_3}}-1\right)\right)^2=\hat{\rho}_2^2.$$

Appendix D

We will prove that $D_{\hat{y}}\mu_1=\hat{\rho}_1\mu_1;\ D_{\hat{y}}\mu_2=\hat{\rho}_2\mu_1+\mu_3;\ D_{\hat{y}}\mu_3=3\hat{\rho}_2\mu_3$.

The first equality is trivial because $D_{\hat{y}}\mu_1=0$ and $\rho_1=0$.

Next, $D_{\hat{y}}\mu_2=\left(D_{\hat{c}}+\frac{s-y_0}{\hat{c}}\sqrt{\frac{s-y_3}{y_0-y_3}}D_s\right)(s-y_0)=\frac{s-y_0}{\hat{c}}\sqrt{\frac{s-y_3}{y_0-y_3}}$. But

$$\rho_2\mu_1+\mu_3=\frac{1}{2\hat{c}}\left(\sqrt{\frac{s-y_3}{y_0-y_3}}-1\right)(s-y_0)+\frac{s-y_0}{2\hat{c}}\left(\sqrt{\frac{s-y_3}{y_0-y_3}}+1\right)=\frac{s-y_0}{\hat{c}}\sqrt{\frac{s-y_3}{y_0-y_3}}.$$

Finally,

$$D_{\hat{y}}\mu_3=\left(D_{\hat{c}}+\frac{s-y_0}{\hat{c}}\sqrt{\frac{s-y_3}{y_0-y_3}}D_s\right)\frac{s-y_0}{2\hat{c}}\left(\sqrt{\frac{s-y_3}{y_0-y_3}}+1\right)$$
$$=-\frac{s-y_0}{2\hat{c}^2}\left(\sqrt{\frac{s-y_3}{y_0-y_3}}+1\right)+\frac{s-y_0}{2\hat{c}^2}\sqrt{\frac{s-y_3}{y_0-y_3}}\left(\sqrt{\frac{s-y_3}{y_0-y_3}}+1+\frac{s-y_0}{2\sqrt{(y_0-y_3)(s-y_3)}}\right)$$
$$=-\frac{s-y_0}{2\hat{c}^2}\left(\sqrt{\frac{s-y_3}{y_0-y_3}}+1\right)+\frac{s-y_0}{2\hat{c}^2}\left(\frac{s-y_3}{y_0-y_3}+\sqrt{\frac{s-y_3}{y_0-y_3}}+\frac{s-y_0}{2(y_0-y_3)}\right)=\frac{s-y_0}{2\hat{c}^2}\left(\frac{s-y_3}{y_0-y_3}-1+\frac{s-y_0}{2(y_0-y_3)}\right)$$
$$=\frac{s-y_0}{2\hat{c}^2}\cdot\frac{-2y_0+2y_3+2s-2y_3+s-y_0}{2(y_0-y_3)}=\frac{3(s-y_0)^2}{4\hat{c}^2(y_0-y_3)}.$$

But, on the other hand,

$$3\hat{\rho}_2\mu_3=3\frac{1}{2\hat{c}}\left(\sqrt{\frac{s-y_3}{y_0-y_3}}-1\right)\frac{s-y_0}{2\hat{c}}\left(\sqrt{\frac{s-y_3}{y_0-y_3}}+1\right)=\frac{3(s-y_0)}{4\hat{c}^2}\left(\frac{s-y_3}{y_0-y_3}-1\right)=\frac{3(s-y_0)^2}{4\hat{c}^2(y_0-y_3)},$$

what concludes the proof.

Appendix E

Proof of the equality in (59).

$\hat{p}_0=s$, therefore:

$$\frac{1}{1!}D_{\hat{y}}s=D_{\hat{y}}(\mu_1+\mu_2)=0+\mu_2\hat{\rho}_2+\mu_3=\mu_10+\mu_2\hat{\rho}_2+\mu_3\binom{1}{1}\hat{\rho}_2^0=\frac{1}{1!}\hat{p}_1.$$

Let the following equality hold true:

$$\frac{1}{j!}D_{\hat{y}}^{j}s = \mu_2\hat{\rho}_2^{j} + \mu_3\binom{j}{1}\hat{\rho}_2^{j-1} = \frac{1}{j!}\hat{p}_j.$$

Then,

$$\frac{1}{(j+1)!}D_{\hat{y}}^{j+1}s = \frac{1}{j+1}D_{\hat{y}}\left(\mu_2\rho_2^{j} + \mu_3\binom{j}{1}\hat{\rho}_2^{j-1}\right) = \frac{1}{j+1}\left((\mu_2\hat{\rho}_2 + \mu_3)\hat{\rho}_2^{j} + \mu_2 j\hat{\rho}_2^{j-1}\hat{\rho}_2^{2} + 3j\hat{\rho}_2\hat{\rho}_2^{j-1} + j\mu_3(j-1)\hat{\rho}_2^{j-2}\hat{\rho}_2^{2}\right)$$
$$= \frac{1}{j+1}\left(\mu_2(j+1)\hat{\rho}_2^{j+1} + (1+3j+j(j-1))\mu_3\hat{\rho}_2^{j}\right) = \mu_2\hat{\rho}_2^{j+1} + \mu_3\binom{j+1}{1}\hat{\rho}_2^{j} = \frac{1}{(j+1)!}\hat{p}_{j+1}$$

what concludes the proof.

Appendix F

$$\begin{aligned}
\hat{y}(z,\hat{c},s) &= \sum_{j=0}^{+\infty}\frac{(z-\hat{c})^{j}}{j!}D_{\hat{y}}^{j}s \\
&= \sum_{j=0}^{+\infty}(z-\hat{c})^{j}\left(y_0\hat{c}^{j} + (s-y_0)\left(\frac{1}{2\hat{c}}\left(\sqrt{\frac{s-y_3}{y_0-y_3}}+1\right)\right)^{j} + \frac{s-y_0}{2\hat{c}}\left(\sqrt{\frac{s-y_3}{y_0-y_3}}+1\right)j\left(\frac{1}{2\hat{c}}\left(\sqrt{\frac{s-y_3}{y_0-y_3}}-1\right)\right)^{j-1}\right) \\
&= y_0 + (s-y_0)\frac{1}{1-\frac{1}{2\hat{c}}\left(\sqrt{\frac{s-y_3}{y_0-y_3}}-1\right)(z-\hat{c})} + \frac{s-y_0}{2\hat{c}}\left(\sqrt{\frac{s-y_3}{y_0-y_3}}+1\right)\frac{(z-\hat{c})}{\left(1-\frac{1}{2\hat{c}}\left(\sqrt{\frac{s-y_3}{y_0-y_3}}-1\right)(z-\hat{c})\right)^{2}} \\
&= y_0 + (s-y_0)\frac{\left(1-\frac{1}{2\hat{c}}\left(\sqrt{\frac{s-y_3}{y_0-y_3}}-1\right)(z-\hat{c}) + (z-\hat{c})\frac{1}{2\hat{c}}\left(\sqrt{\frac{s-y_3}{y_0-y_3}}+1\right)\right)}{\left(1-\frac{1}{2\hat{c}}\left(\sqrt{\frac{s-y_3}{y_0-y_3}}-1\right)(z-\hat{c})\right)^{2}} \\
&= y_0 + (s-y_0)\frac{\hat{c} + (z-\hat{c})4\hat{c}(y_0-y_3)}{\left(2\hat{c}\sqrt{y_0-y_3} - \left(\sqrt{s-y_3}-\sqrt{y_0-y_3}\right)(z-\hat{c})\right)^{2}} \\
&= y_0 + \frac{4(s-y_0)(y_0-y_3)z\hat{c}}{\left(\left(\sqrt{s-y_3}+\sqrt{y_0-y_3}\right)\hat{c} - \left(\sqrt{s-y_3}-\sqrt{y_0-y_3}\right)z\right)^{2}}.
\end{aligned}$$

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