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H-rank as a Control Tool for Discrete Dynamical Systems

Mantas Landauskas, Jurate Ragulskiene and Minvydas Ragulskis

Research Group for Mathematical and Numerical Analysis of Dynamical Systems, Kaunas University of Technology, Studentu 50-222, Kaunas, LT-51368, Lithuania

Abstract. The convergence to Arnold tongues is studied using computational techniques based on ranks of Hankel matrices (H-ranks). The ranks of Hankel matrices carry important physical information about transient processes taking place in discrete nonlinear iterative maps. We will show that the measurements of the convergence rate to Arlond tongues can reveal important physical information on the properties of the iterative system. Moreover, such enriched representation of Arnold tongues produces aesthetically beautiful pictures.

Keywords: Hankel matrix, Circle map
PACS: 89.70.Eg, 89.90.+n, 89.75.Kd

INTRODUCTION

Computation and visualization of H-ranks in the space of system's parameters and initial conditions provides the insight into the embedded algebraic complexity of the nonlinear system. It is shown in [1] that the computation of H-ranks can be effectively used to identify and assess the sensitivity of nonlinear systems to initial conditions and can be used as a simple and effective numerical tool for qualitative investigation of the onset of chaos for discrete nonlinear iterative maps. But the main purpose of this paper is to show the use of H-rank as a control tool for a discrete dynamical system.

Numerical convergence of the discrete logistic map gauged with a finite computational accuracy is investigated in [2] where forward iterations are used to identify self-similar patterns in the region before the onset to chaos. An alternative technique based on the concept of the H-rank is proposed in [1] for clocking the convergence of iterative chaotic maps. The H-rank also reveals three manifolds of the discrete iterative map: the stable manifold, the unstable manifold and the manifold of the non-asymptotic convergence although this is out of the scope of this paper.

The main purpose of this paper is to show the use of H-rank as a control tool for a discrete dynamical system. We will use the discrete iterative circle map to illustrate the process of convergence to stationary states. The circle map is exploited in numerous models of nonlinear dynamical systems whenever the effects of quasiperiodicity are encountered [4, 5, 6, 7]. We will show that the study of the convergence rate to a periodic orbit of a single circle map can produce the patterns which contain important information on the stability of periodic orbits of the circle map. This information could be useful whenever the manipulation or control of quasiperiodic nonlinear systems would be considered.

THE ALGORITHM FOR THE COMPUTATION OF THE H-RANK

The concept of the H-rank of a sequence $(p_j; j = 0, 1, \ldots)$; $p_j \in \mathbf{R}$ has been introduced in [1]. The main purpose of this section is to investigate the properties of the H-rank of a solution of a discrete iterative map when transient processes are omitted and the system approaches a periodic or a quasiperiodic orbit.

A sequence of Hankel matrices reads:

$$H_n := \left(p_{i+j-2}\right)_{1\le i,j\le n} = \begin{bmatrix} p_0 & p_1 & \cdots & p_{n-1} \\ p_1 & p_2 & \cdots & p_n \\ \cdots & \cdots & \cdots & \cdots \\ p_{n-1} & p_n & \cdots & p_{2n-2} \end{bmatrix};$$

$n = 1, 2, \ldots$. The Hankel transform (the sequence of determinants of Hankel matrices) $(d_n; n = 0, 1, \ldots)$ reads:

$$d_n := detH_n; n = 1, 2, \ldots.$$

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The sequence $(p_j; j = 0, 1, \ldots)$ has an H-rank $m \in \mathbf{Z}_0$; $m < +\infty$; $Hr(p_j; j = 0, 1, \ldots) = m$ if the sequence of determinants of Hankel matrices has the following structure: $(d_1, d_2, \ldots, d_m, 0, 0, \ldots)$ where $d_m \neq 0$ and $d_{m+1} = d_{m+2} = \ldots = 0$.

It is admited that $Hr(0, 0, 0, \ldots) = 0$. Note that $Hr(p_0, \ldots, p_m, 0, 0, 0, \ldots) = m + 1$, if only $p_m \neq 0$ for $m = 0, 1, 2, \ldots$.

It could be proven that the Hankel rank of a solution of a discrete iterative map can only decrease if a part or the whole transient process from the initial condition is omitted [1].

THE CIRCLE MAP

The circle map is represented by the one-dimensional iterative map:

$$\theta_{n+1} = f(\theta_n) = \theta_n + \Omega - \frac{K}{2\pi} \cdot sin(2\pi\theta_n); \tag{1}$$

where θ is a polar angle ($\theta \in [0; 1)$); K is the coupling strength; Ω is the driving phase and $n = 0, 1, \ldots$. For small to intermediate values of K ($0 < K < 1$), and certain values of Ω, the circle map exhibits a phenomenon called phase locking. In a phase-locked region, the values θ_n advance essentially as a rational multiple of n.

The phase-locked regions in $\Omega - K$ parameter plane are called Arnold tongues [4]. Arnold tongues are observed in a variety of nonlinear physical models whenever the effect of phase locking occurs in those systems [3, 4, 5, 6, 7].

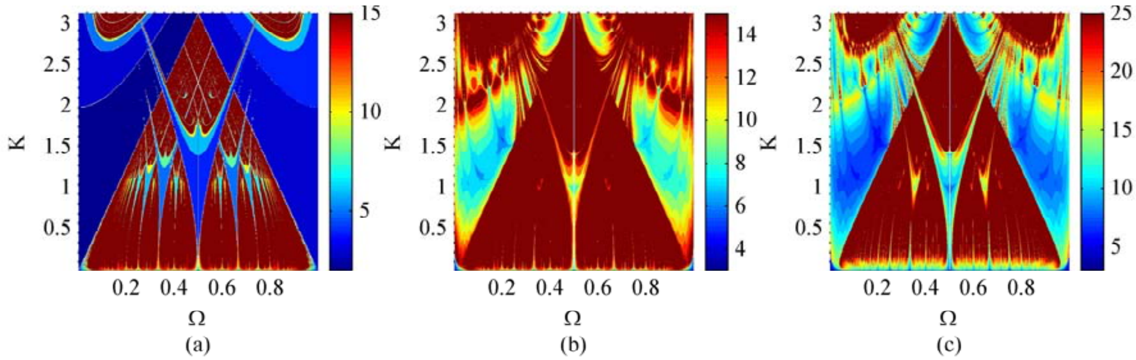

FIGURE 1. Pseudoranks of the circle map. Part (a) was constructed with omitting transient processes and the initial phase was set to $\theta_0 = 0.5$; parts (b) and (c) was constructed without omitting transient processes.

We will use the H-rank as the computational tool for the reconstruction of Arnold tongues. We compute H-ranks in the region $0 \leq \Omega \leq 1$ and $0 \leq K \leq \pi$. For every pair of Ω and K we start the iterative process, omit $k = 4000$ forward iterations (all transients cease down during that period of time), construct the sequence $(\theta_{j+k}; j = 0, 1, \ldots)$ and calculate the H-rank of that sequence. As shown in [1], the H-rank of a chaotic sequence does not exist (it tends to infinity then). Therefore we set the upper limit for the H-rank $\overline{m} = 15$. If the sequence of determinants does not vanish until $m = 15$ we terminate the process assuming that $Hr(p_{j+k}; j = 0, 1, \ldots) = \overline{m}$. The results are shown in Fig. 1(a). Zones where the phase locking occurs are clearly separated by red insets; the produced picture reveals the well-known shape of Arnold tongues in the circle map [5].

We repeat the computational experiment without omitting transient processes ($k = 0$). The initial condition θ_0 is set to 0.5; the results shows that the H-rank of the process with transient processes is higher compared to the the H-rank of the same process but with transient processess omitted (Fig. 1(b)). The higher value of the upper bound ($\overline{m} = 25$) helps to visualize complex transient processes in regions where phase locking occurs (Fig. 1)(c). The interesting pattern in the blue colored zones (corresponding to phase-locked regions in the $\Omega - K$ plane) is rather unexpected and requires additional attention.

It is natural to expect that the initial condition $\theta_0 = \theta^\star = 0.135$ should yield a minimal rank at $\Omega = 0.15$ and $K = 0.4\pi$. We repeat computational experiment by constructing the map of pseudoranks when the initial phase is set to $\theta_0 = 0.135$ which also showed to be a clear illustration of the complex dynamic behavior of the circle map. The convergence process to the phase-locked behavior (Arnold tongues) is far from trivial and one does need to pay careful attention to many circumstances whenever the generation or control of phase-locked regimes in discrete iterative maps is considered.

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Though theoretically one needs to find such matrix dimension that the determinant of the Hankel matrix is equal to zero, in practice it suffices to compute determinants up to a certain precision, like the machine epsilon. Thus we continue the computation of determinants until $\left|\tilde{H}_m\right| < \varepsilon$. In this respect our computations reveal not the rank, but the pseudorank of a sequence.

The selection of a particular value of ε requires additional attention. The structure of Arnold tongues in the circle map is well-known thus we select Fig. 1(a), fix the parameter K ($K = 0.4\pi$) and cut through the map of pseudoranks by varying the parameter Ω. Moreover, we perform the computation of pseudoranks for different initial conditions ($0 \leq \theta_0 \leq 1$) for different ε. We fix $\varepsilon = 10^{-30}$ and use this value for the computation of pseudoranks (all previous figures of pseudoranks are constructed using this value of ε).

Let us consider the system to be in the regime where $\Omega = 0.35$, $K = 2$ and $\theta_0 = 0.135$. Say the map converges to a particular fixed point. While being able to control the parameter Ω we want the system to fall into the stable period-1 regime with the transient process to be as long as possible.

Now from the condition $f(\theta_{n+1}) = f(\theta_n)$ it follows that $\Omega = \frac{K}{2\pi} sin(2\pi\theta_n)$. This holds true if $\Omega^\star \leq \frac{K}{2\pi}$. Otherwise the map does not converge to a stable point. In this particular case the stable period-1 regime exists if $\Omega^\star \leq \frac{1}{\pi}$. By replacing the particular Ω with $\Omega^\star = \frac{K}{2\pi} sin(2\pi\theta_n)$ the system suddenly falls in the stable mode. But in this case the duration of the transient process is equal to zero. Thus another $\Omega^\star \leq 0.1$ must be considered. And this is the value which represents the highest rank of the sequence in the phase plane $\Omega - K$.

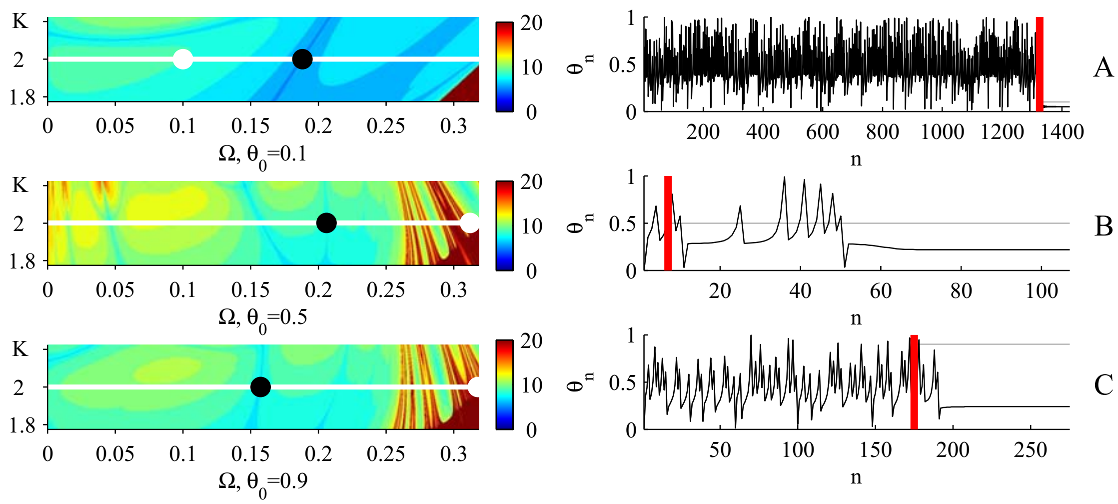

FIGURE 2. The adjusting of $\Omega^\star$. The black dots shows the lowest ranks at $K = 2$, the white dots shows the highest ranks at the same K.

Numerical experiments we performed enables one to take control of the particular system defined by a circle map. Fig. 2 illustrates the algorithm of finding the control function $g(\theta_0)$. The value of this function is the value of $\Omega^\star$ one needs to set to Ω at the step n (or the moment n) in order to drive the system to period-1 stable regime. The argument of the function is equal to θ_n. Thus θ_m, $m > n + n_0$, becomes equal after a transient process as that was the purpose. Here n_0 is the number of steps this transient process takes. Due to the mentioned technique the duration of the transient process is maximized.

It must be noted that by changing Ω in particular circle map we are affecting its behavior. That is the fixed point changes. Or in other words: we modify the system to slowly converge to another stable state. It is reasonable to perform the mentioned correction of Ω after detecting the system becoming stationary. Otherwise the correction might result in shorter transient process than that of original system. Furthermore, such control technique enables one to keep the system running. And if one finds the $\Omega^\star \geq \frac{K}{2\pi}$ with the highest rank the system might be revived once and for all because there might be no fixed point.

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CONCLUDING REMARKS

The convergence to Arnold tongues is studied in this paper. The existence of phase-locked regions in the circle map is a well-known and explored topic in the area of discrete nonlinear iterative maps. But it appears that the process of convergence to Arnold tongues is far from being trivial. Such complexity of transient processes poses an important feature for the applicability of control techniques applicable for the circle map.

Let us consider a situation when the circle map is operating in a chaotic regime and one needs to bring it to the phase-locked regime (assume that there is a possibility to control one or both parameters of the circle map). One may jump from the chaotic region into the phase-locked region in the parameter plane of the circle map in one forward iteration. But the convergence to the actual phase-locked regime depends on the instantaneous phase of the system right after the jump. One can determine the phase of the circle map immediately before the jump and adjust parameters of the map in such a way that the system is placed onto the manifold of non-asymptotic convergence immediately after the adjustment of the parameters is performed. Such control method would ensure that the system will start operating in the phase-locked mode immediately after the correction of system's parameters. Otherwise (if the distribution of manifolds on Arnold tongues is unknown) a long transient process could be required until the system starts operating in the phase-locked mode.

The development of such control strategies is a definite object of future research. But pictures representing the manifold of non-asymptotic convergence do carry not only an important physical information but are also aesthetically beautiful.

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