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Order adaptive integration rule with equivalently weighted internal nodes

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Abstract

Purpose – To develop order adaptive integration rule without limitation requiring that the number of equally spaced nodes must be a divisible numeral. Such integration technique could be of great practical value for different engineering applications where partition adaptability is impossible and use of standard high order integration techniques is unfeasible due to the fact that a significant number of nodes at the end of the sampling sequence must be deleted until the needed divisibility of the number of nodes is achieved.

Design/methodology/approach – Finite element approximation is used for the subdivision of the domain of integration and the development of order adaptive integration rule.

Findings – New integration rule is developed. It has a number of interesting features. Weights of the internal nodes are equivalent and equal to one. That makes the computational implementation of the integration rule very easy. Weights not equal to one are located only at the beginning and at the end of the sequence and are symmetric. For an m-th order rule the number of weights not equal to one is $2m$ if m is odd.

Originality/value – For different engineering applications where the integration order can be controlled without changing the number of nodes, especially for real time applications where the number of discrete samples is unknown before the experiment.

Keywords Numerical control, Finite element analysis, Adaptability

Paper type Technical paper

Introduction

A definite integral:

$$I(f) := \int_a^b f(t)\,\mathrm{d}t$$

can be approximated by an integration rule:

$$R(f) = \sum_{i=1}^{n} \omega_i f(t_i)$$

where t_i are the nodes and ω_i are the weights of the n-point integration rule R. Automatic algorithms are widely used for the numerical approximation of definite integrals. A quadrature routine is adaptive if the nodes and the weights are chosen in a way that depends on the integrand. Adaptive integration algorithms can be order adaptive or partition adaptive.

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In an order adaptive algorithm a sequence of increasingly higher order rules is applied on the integration interval. Such algorithms cope well with smooth integrands, without peaks and singularities. Partition adaptive algorithms manipulate with a partition of the interval maintaining fixed integration rule applied to each subinterval. Doubly adaptive algorithms are both order and partition adaptive. A survey of automatic integration routines is available in Davis and Rabinowitz (1984), Berntsen and Espelid (1991) and Venter and Laurie (2002).

It is quite common in engineering applications that the sampling process of the integrand is rather costly. A typical example is time average laser holography where the surface of a vibrating structure is illuminated using a stroboscopic laser. The intensity of illumination in the plane of the hologram is then defined by a definite integral over the exposure time (Vest, 1979).

As optical fringe patterns are sometimes hard to interpret, hybrid numerical – experimental techniques are widely used for inverse engineering problems (Holstein *et al.*, 2001; Rhee and Rowlands, 2002). Accurate numerical reconstruction of fringe patterns generated by stroboscopic laser holography is rather complicated, especially when the vibration of the analyzed structure is not stationary.

As the pulse rate of the laser is predefined before the experiment, partition adaptability cannot be used (nodes t_i cannot be altered). The only way to increase the accuracy of the reconstructed images is raising the order of the rules applied on the predefined integration interval. Unfortunately, third-order integration rule (Simplex rule) Davis and Rabinowitz (1984) already requires that the number of nodes must be odd. Higher-order Newton-Kotes quadrature formulas (Davis and Rabinowitz, 1984) require that the number of nodes is divisible even from higher numerals. For post-processing applications it means that a significant number of nodes at the end of the experimental sequence must be deleted and the integration interval (exposure time) artificially shortened.

Therefore, there exists a definite need for an order adaptive integration rule without a demand that the number of nodes must be divisible. The proposed integration rule satisfies this requirement. Moreover, the weights of the internal nodes are equal to 1, what makes the implementation of the rule very easy. Such integration rule could be useful in many different engineering applications when the time step is constant and the number of time steps is not known at the beginning of the experiment.

Illustrative example

The procedure of the derivation of the integration rule is illustrated by the following example.

A one-dimensional finite Lagrange element (Bathe, 1982) with three nodes is considered. Let the local co-ordinates ζ of the nodes be -1 (node 1), 0 (node 2) and 1 (node 3).

The shape functions N_i, $i = 1, 2, 3$ are constructed as Lagrange interpolation functions, satisfying the following nodal conditions (Bathe, 1982):

$$\begin{aligned} &N_1(-1) = N_2(0) = N_3(1) = 1; \\ &N_1(0) = N_1(1) = N_2(-1) = N_2(1) = N_3(-1) = N_3(0) = 0. \end{aligned} \tag{1}$$

Thus:

$$N_1(\zeta) = \frac{\zeta^2}{2} - \frac{\zeta}{2}; \quad N_2(\zeta) = -\zeta^2 + 1; \quad N_3(\zeta) = \frac{\zeta^2}{2} + \frac{\zeta}{2}. \tag{2}$$

A continuous function can be interpolated in the domain of the finite element through the nodal values of that function (f_k at node 1; f_{k+1} at node 2 and f_{k+2} at node 3):

$$\hat{f}(\zeta) = f_k N_1(\zeta) + f_{k+1} N_2(\zeta) + f_{k+2} N_3(\zeta). \tag{3}$$

It is clear that the interpolated function $\hat{f}$ is a polynomial of the second-order.

Let's assume that a function f is sampled at equally spaced time steps starting from t_0 and producing eight discrete function values f_i, $i = 1, \ldots, 8$ (seven time steps); the length of a time step is h. The first finite element E_1 is positioned in such a way that its first node is located at t_0; the second node – at $t_0 + h$; the third node – at $t_0 + 2h$ (Figure 1). Every next finite element E_k, $k = 2, \ldots, 6$ is shifted with respect to E_{k-1} by one interval h.

Integral in the global domain $t_0 \le t \le t_0 + 7h$ is calculated in the following way (Figure 1):

$$\begin{aligned}\int_{t_0}^{t_0+7h} f\,\mathrm{d}t = \int_{t_0}^{t_0+(3/2)h} f(t)\,\mathrm{d}t + \int_{t_0+(3/2)h}^{t_0+(5/2)h} f(t)\,\mathrm{d}t + \cdots + \int_{t_0+(9/2)h}^{t_0+(11/2)h} f(t)\,\mathrm{d}t \\ + \int_{t_0+(11/2)h}^{t_0+7h} f(t)\,\mathrm{d}t.\end{aligned} \tag{4}$$

Introduction of the variable $t = t_0 + h(\zeta + 1)$ and the assumption that the function f is a second-order polynomial leads to the following expression of the first integral on the right side of equation (4):

$$\begin{aligned}\int_{t_0}^{t_0+(3/2)h} f(t)\,\mathrm{d}t &= h\int_{-1}^{0.5} f(\zeta)\mathrm{d}\zeta \\ &= h\left(f_1\int_{-1}^{0.5} N_1(\zeta)\mathrm{d}\zeta + f_2\int_{-1}^{0.5} N_2(\zeta)\mathrm{d}\zeta + f_3\int_{-1}^{0.5} N_3(\zeta)\,\mathrm{d}\zeta\right)\end{aligned} \tag{5}$$

Now, if $t = t_0 + h(\zeta + 2)$:

$$\begin{aligned}\int_{t_0+(3/2)h}^{t_0+(5/2)h} f(t)\,\mathrm{d}t &= h\int_{-0.5}^{0.5} f(\zeta)\mathrm{d}\zeta \\ &= h\left(f_2\int_{-0.5}^{0.5} N_1(\zeta)\mathrm{d}\zeta + f_3\int_{-0.5}^{0.5} N_2(\zeta)\mathrm{d}\zeta + f_4\int_{-0.5}^{0.5} N_3(\zeta)\mathrm{d}\zeta\right)\end{aligned} \tag{6}$$

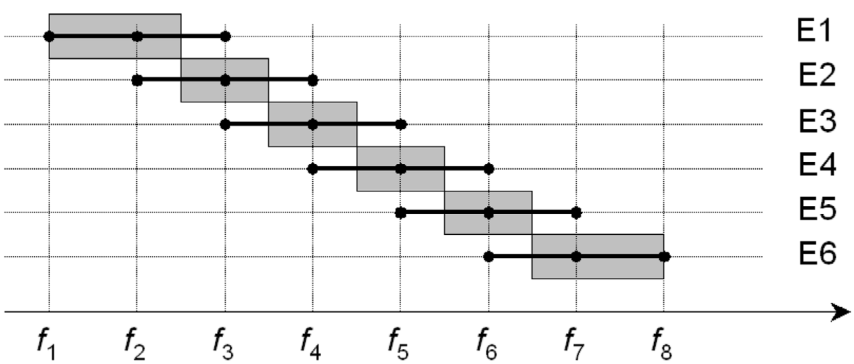

Figure 1.
Direct stiffness procedure at $m = 3$; $n = 8$

Similar procedure can be continued with all middle integrals until the last one, where $t = t_0 + h(\zeta + 6)$:

$$\begin{aligned}\int_{t_0+(11/2)h}^{t_0+7h} f(t)\,\mathrm{d}t &= h\int_{-0.5}^{1} f(\zeta)\mathrm{d}\zeta \\ &= h\left(f_6\int_{-0.5}^{1} N_1(\zeta)\mathrm{d}\zeta + f_7\int_{-0.5}^{1} N_2(\zeta)\mathrm{d}\zeta + f_8\int_{-0.5}^{1} N_3(\zeta)\mathrm{d}\zeta\right)\end{aligned} \tag{7}$$

The following integrals can be calculated explicitly:

$$\begin{aligned}&\int_{-1}^{0.5} N_1(\zeta)\mathrm{d}\zeta = \frac{3}{8};\quad \int_{-1}^{0.5} N_2(\zeta)\mathrm{d}\zeta = \frac{9}{8};\quad \int_{-1}^{0.5} N_3(\zeta)\mathrm{d}\zeta = 0 \\ &\int_{-0.5}^{0.5} N_1(\zeta)\mathrm{d}\zeta = \frac{1}{24};\quad \int_{-0.5}^{0.5} N_2(\zeta)\mathrm{d}\zeta = \frac{11}{12};\quad \int_{-0.5}^{0.5} N_3(\zeta)\mathrm{d}\zeta = \frac{1}{24}; \\ &\int_{-0.5}^{1} N_1(\zeta)\mathrm{d}\zeta = 0;\quad \int_{-0.5}^{1} N_2(\zeta)\mathrm{d}\zeta = \frac{9}{8};\quad \int_{-0.5}^{1} N_3(\zeta)\mathrm{d}\zeta = \frac{3}{8}.\end{aligned} \tag{8}$$

Collection of terms at f_i in equation (4) results into a direct stiffness procedure (Bathe, 1982) executed for all finite elements E_k; $k = 1, \ldots, 6$ (Table I). Thus:

$$\begin{aligned}\int_{t_0}^{t_0+7h} f(t)\,\mathrm{d}t &= h\left(\frac{3}{8}f_1 + \frac{7}{6}f_2 + \frac{23}{24}f_3 + f_4 + f_5 + \frac{23}{24}f_6 + \frac{7}{6}f_7 + \frac{3}{8}f_8\right) \\ &= h(a_1f_1 + a_2f_2 + a_3f_3 + f_4 + f_5 + a_3f_6 + a_2f_7 + a_1f_8)\end{aligned} \tag{9}$$

It can be noted that $a_1 + a_2 + a_3 = 3/8 + 7/6 + 23/24 = 2.5$.

Important is the fact that calculated coefficients a_i can be used in integration formulas with different numbers of time steps. For example, if the number of time steps is 8:

$$\int_{t_0}^{t_0+8h} f(t)\,\mathrm{d}t = h(a_1f_1 + a_2f_2 + a_3f_3 + f_4 + f_5 + f_6 + a_3f_7 + a_2f_8 + a_1f_9) \tag{10}$$

where $f_i = f(t_0 + h(i-1))$; $i = 1, \ldots, 9$.

I	1	2	3	4	5	6	7	8
E1	3/8	9/8	0					
E2		1/24	11/12	1/24				
E3			1/24	11/12	1/24			
E4				1/24	11/12	1/24		
E5					1/24	11/12	1/24	
E6						0	9/8	3/8
Σ	3/8	7/6	23/24	1	1	23/24	7/6	3/8
a_i	3/8	7/6	23/24					

Table I. Derivation of coefficients a_i at $m = 3$; $n = 8$

It is clear that the minimum number of discrete function values f_i needed to guarantee that the second-order polynomials are integrated exactly with the derived coefficients a_i is 6.

General case

Lagrange finite elements with m nodes are used for interpolation of nodal values of a discrete function in the domain of a finite element. Let the number of discrete time moments where the function values f_i are sampled be n ($n \geq 2m$); time step is h.

The first finite element E_1 is located as follows – first node at t_0; second node at $t_0 + h$; last node at $t_0 + (m-1)h$. The second finite element E_2 is shifted with respect to E_1 by h – its first node is located at $t_0 + h$; second node at $t_0 + 2h$; last node at $t_0 + mh$. The process is continued until the last finite element E_{n-m+1} is placed as follows – first node at $t_0 + (n-m)h$; second node at $t_0 + (n-m+1)h$; last node at $t_0 + (n-1)h$.

Let a local co-ordinate ζ be assigned for every Lagrange finite element so that the co-ordinate of the i-th node of a finite element is $-1 + 2(i-1)/(m-1)$; $i = 1, \ldots, m$. Nodal shape functions are constructed as Lagrange functions (Bathe, 1982) satisfying the conditions:

$$N_i\left(-1+2\frac{i-1}{m-1}\right)=1;\; N_i\left(-1+2\frac{j-1}{m-1}\right)=0;\; i=1,\ldots,m;\, j=1,\ldots,m;\, j\neq i \quad (11)$$

A continuous function can be interpolated in the domain of the finite element through the nodal values of that function (f_k at node 1; f_{k+1} at node 2 and f_{k+m-1} at node m):

$$\hat{f}(\zeta) = f_k N_1(\zeta) + f_{k+1} N_2(\zeta) + \cdots + f_{k+m-1} N_m(\zeta); \quad -1 \leq \zeta \leq 1 \quad (12)$$

It is clear that the interpolated function is a polynomial of the $(m-1)$ order.

The integral of the function $f(t)$ is subdivided into $n - m + 1$ integrals:

$$\begin{aligned}\int_{t_0}^{t_0+(n-1)h} f(t)\,\mathrm{d}t = \int_{t_0}^{t_0+(hm/2)} f(t)\,\mathrm{d}t + \sum_{s=2}^{n-m}\left(\int_{t_0+(hm/2)+(s-2)h}^{t_0+(hm/2)+(s-1)h} f(t)\,\mathrm{d}t\right) \\ + \int_{t_0+(hm/2)+(n-m-1)h}^{t_0+(n-1)h} f(t)\,\mathrm{d}t\end{aligned} \quad (13)$$

Introduction of the variable $t = t_0 + 0.5h(m-1)(\zeta + 1)$ and the assumption that the function f is a polynomial of the $(m-1)$-st order produce the following result:

$$\begin{aligned}\int_{t_0}^{t_0+(hm/2)} f(t)\,\mathrm{d}t &= \frac{h(m-1)}{2}\int_{-1}^{1/(m-1)} f(\zeta)\,\mathrm{d}\zeta \\ &= \frac{h(m-1)}{2}\sum_{i=1}^{m} f_i \int_{-1}^{1/(m-1)} N_i(\zeta)\,\mathrm{d}\zeta\end{aligned} \quad (14)$$

Analogously, $t = t_0 + h(s-1) + 0.5h(m-1)(\zeta + 1)$; $s = 2, \ldots, (n-m)$ produces the following results:

$$\int_{t_0+(hm/2)+(s-2)h}^{t_0+(hm/2)+(s-1)h} f(t)\,\mathrm{d}t = \frac{h(m-1)}{2}\int_{-1/(m-1)}^{1/(m-1)} f(\zeta)\,\mathrm{d}\zeta = \frac{h(m-1)}{2}\sum_{i=1}^{m} f_{i+s-1}\int_{-1/(m-1)}^{1/(m-1)} N_i(\zeta)\,\mathrm{d}\zeta \tag{15}$$

Finally, at $t = t_0 + h(n-m) + 0.5h(m-1)(\zeta+1)$:

$$\int_{t_0+(hm/2)+(n-m-1)h}^{t_0+(n-1)h} f(t)\,\mathrm{d}t = \frac{h(m-1)}{2}\int_{-1/(m-1)}^{1} f(\zeta)\,\mathrm{d}\zeta = \frac{h(m-1)}{2}\sum_{i=1}^{m} f_{i+n-m}\int_{-1/(m-1)}^{1} N_i(\zeta)\,\mathrm{d}\zeta \tag{16}$$

Next we introduce the following notations:

$$\begin{aligned} n_i^L &= \int_{-1}^{1/(m-1)} N_i(\zeta)\mathrm{d}\zeta; \\ n_i^C &= \int_{-1/(m-1)}^{1/(m-1)} N_i(\zeta)\mathrm{d}\zeta; \\ n_i^R &= \int_{-1/(m-1)}^{1} N_i(\zeta)d\zeta; \quad i = 1, \ldots, m \end{aligned} \tag{17}$$

Then:

$$\begin{aligned} \int_{t_0}^{t_0+(n-1)h} f(t)\,\mathrm{d}t = \frac{h(m-1)}{2}\Bigg[& f_1 n_1^L + f_2\left(n_2^L + n_1^C\right) + \cdots + f_m\left(n_m^L + \sum_{i=1}^{m-1} n_{m-i}^C\right) \\ & + \sum_{j=1}^{n-2m} f_{m+j}\sum_{i=1}^{m} n_i^C + f_{n-m+1}\left(n_1^R + \sum_{i=1}^{m-1} n_{m-i+1}^C\right) \\ & + \cdots + f_{n-1}\left(n_{m-1}^R + n_m^C\right) + f_n n_m^R \Bigg] \end{aligned} \tag{18}$$

It can be noted that the shape functions $N_i(\zeta)$ are available in standard finite element software codes (Bathe, 1982). Explicitly the shape functions can be expressed as $(m-1)$-st order polynomials:

$$N_i(\zeta) = c_{i0} + c_{i1}\zeta + \cdots + c_{i(m-1)}\zeta^{m-1}; \quad i = 1, \ldots, m \tag{19}$$

and the unknown coefficients can be determined from the following system of linear algebraic equations which follows from equation (11):

$$\begin{bmatrix} c_{10} & \cdots & c_{1(m-1)} \\ \vdots & & \vdots \\ c_{m0} & \cdots & c_{m(m-1)} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ -1 & -1+\frac{2}{m-1} & -1+\frac{4}{m-1} & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ (-1)^{m-1} & \left(-1+\frac{2}{m-1}\right)^{m-1} & \left(-1+\frac{4}{m-1}\right)^{m-1} & \cdots & 1 \end{bmatrix} = E \quad (20)$$

where E is the identity matrix.

As the local coordinates of the first and the last nodes of a Lagrange finite element are -1 and 1 accordingly, its shape functions $N_i(\zeta)$ and $N_{m-i+1}(\zeta)$ are symmetric to each other with respect to the axis $\zeta = 0$ for all $i = 1, \ldots, m$ (Bathe, 1982). Then:

$$n_i^L = n_{m-i+1}^R; \quad n_i^C = n_{m-i+1}^C; \quad i = 1, \ldots, m \quad (21)$$

Thus, the coefficients at f_i and f_{m-i+1} (equation (18)) are equal for all $i = 1, \ldots, m$.

We denote:

$$\begin{aligned} a_1 &= \frac{m-1}{2} n_1^L; \quad a_2 = \frac{m-1}{2}\left(n_2^L + n_1^C\right); \ldots; \\ a_m &= \frac{m-1}{2}\left(n_m^L + \sum_{i=1}^{m-1} n_{m-i}^C\right); \quad a_0 = \frac{m-1}{2} \sum_{i=1}^{m} n_i^C \end{aligned} \quad (22)$$

Thus:

$$\begin{aligned} \int_{t_0}^{t_0+(n-1)h} f(t)\,\mathrm{d}t = h\Big(&a_1 f_1 + a_2 f_2 + \cdots + a_m f_m \\ &+ \sum_{i=1}^{n-2m} a_0 f_{m+i} + a_m f_{n-m+1} + \cdots + a_2 f_{n-1} + a_1 f_n\Big) \end{aligned} \quad (23)$$

Lets assume that $f(t) = C = \text{const}$ and $n > 2m$. Then:

$$C\int_{t_0}^{t_0+nh} \mathrm{d}t - C\int_{t_0}^{t_0+(n-1)h} \mathrm{d}t = hC \quad (24)$$

But from equation (23) it follows that:

$$\begin{aligned} C\int_{t_0}^{t_0+nh} \mathrm{d}t - C\int_{t_0}^{t_0+(n-1)h} \mathrm{d}t &= hC\left(2\sum_{i=1}^{m} a_i + (n-2m+1)a_0\right) \\ &\quad - hC\left(2\sum_{i=1}^{m} a_i + (n-2m)a_0\right) \\ &= a_0 hC \end{aligned} \quad (25)$$

Therefore, $a_0 = 1$. As the calculation of a_0 is independent from the nodal values of $f(t)$ it follows that $a_0 = 1$ for any f_i.

On the other hand:

$$C\int_{t_0}^{t_0+(n-1)h}\mathrm{d}t = hC\left(2\sum_{i=1}^{m} a_i + n - 2m\right) = hC(n-1) \tag{26}$$

Thus:

$$\sum_{i=1}^{m} a_i = m - \frac{1}{2} \tag{27}$$

Finally:

$$\int_{t_0}^{t_0+(n-1)h} f(t)\mathrm{d}t = h\left(\sum_{i=1}^{m} a_i f_i + \sum_{i=1}^{n-2m} f_{m+i} + \sum_{i=1}^{m} a_{m-i+1} f_{n-m+i}\right) \tag{28}$$

It is clear that the derived integration rule is exact when the integrated function is a polynomial of the $(m-1)$-st order. We will prove that the derived integration rule is exact when the integrated function is a polynomial of the m-th order if m is odd.

Coefficients a_i used in equation (28) are listed in Table II at different m; parameter p denotes the maximum degree of exactly integrated polynomial.

Lemma. Integration rule presented in equation (28) is exact when $f(t)$ is a polynomial of the m-th order if m is odd.

Proof. One and only polynomial of the $(m-1)$-st order can be interpolated in the domain of the i-th finite element through the discrete values of the function (Figure 2):

$$\hat{f}_i(t) = r_{i,0} + r_{i,1}t + \cdots + r_{i,(m-1)}t^{m-1} \tag{29}$$

Many different polynomials of the m-th order can be interpolated through the same points:

$$P_i(t) = s_{i,0} + s_{i,1}t + \cdots + s_{i,m}t^{m} \tag{30}$$

Coefficients r and s in equations (29) and (30) are real numbers. A difference between these polynomials is calculated as:

$$\Delta_i(t) = P_i(t) - \hat{f}_i(t) \tag{31}$$

As both polynomials interpolate the same points, the abscises of those points are the roots of $\Delta_i(t)$:

$$\Delta_i(t_0 + jh) = 0 \quad \text{at } j = (i-1),\ i, \ldots, (i+m-2). \tag{32}$$

Then, in accordance to the Viete theorem (James, 2003), $\Delta_i(t)$ can be expressed like:

$$\Delta_i(t) = s_{i,m}[t - t_0 - (i-1)h][t - t_0 - ih]\cdots[t - t_0 - (i+m-2)h] \tag{33}$$

Introduction of new variable:

$$\zeta_i = \frac{2}{h(m-1)}\left(t - t_0 - \left(i + \frac{m-3}{2}\right)h\right)$$

helps to simplify the expression of $\Delta_i(t)$:

Table II.
Coefficients of the integration formulas with equivalently weighted internal nodes

m	2	3	4	5	6	7	8	9	10	11
a_1	0.5	0.375	0.333333	0.329861	0.31875	0.304225	0.29418	0.286975	0.280344	0.274266
a_2	1	1.16667	1.29167	1.32083	1.37639	1.46038	1.5307	1.58902	1.6487	1.70933
a_3		0.958333	0.833333	0.766667	0.655556	0.453464	0.242526	0.0359852	−0.202745	−0.474888
a_4			1.04167	1.10139	1.2125	1.47143	1.82299	2.24089	2.79793	3.52179
a_5				0.98125	0.925694	0.739393	0.387831	−0.140564	−0.97612	−2.23963
a_6					1.01111	1.08247	1.29341	1.72094	2.5565	4.06882
a_7						0.988633	0.91832	0.702145	0.145108	−1.11192
a_8							1.01004	1.0725	1.31123	2.02767
a_9								0.992107	0.932425	0.664452
a_{10}									1.00663	1.06603
a_{11}										0.994076
p	1	3	3	5	5	7	7	9	9	11
$\Sigma\ a_i$	1.5	2.5	3.5	4.5	5.5	6.5	7.5	8.5	9.5	10.5

$$\Delta_i(\zeta_i) = s_{i,m}(\zeta_i+1)\left(\zeta_i+\frac{m-3}{m-1}\right)\ldots\zeta_i\ldots(\zeta_i-1) \tag{34}$$

Then the integral of $\Delta_i(t)$ in the domain of the i-th finite element is:

$$\int_{t_0+(i-1)h}^{t_0+(i+m-2)h}\Delta_i(t)\,\mathrm{d}t = \frac{h(m-1)}{2}\int_{-1}^{1}\Delta_i(\zeta_i)\mathrm{d}\zeta_i = 0 \tag{35}$$

The integral is equal to 0 because the integrand is an odd function and the limits of the integral are symmetric.

The limits of the integrals in the domains of the internal finite elements E_i, $i = 2, 3, \ldots, (n-m)$ are $(-1/(m-1)) \le \zeta_i \le (1/(m-1))$ (equation (15)). These integrals are also equal to zero due to the symmetry of limits.

For the first and the last finite elements the limits of integration are $-1 \le \zeta_1 \le 1/(m-1)$ and $(-1/(m-1)) \le \zeta_{n-m+1} \le 1$ accordingly. Then:

$$\begin{aligned}&\int_{-1}^{1/(m-1)}\Delta_1(\zeta_1)\mathrm{d}\zeta_1 + \int_{-1/(m-1)}^{1}\Delta_{n-m+1}(\zeta_{n-m+1})\mathrm{d}\zeta_{n-m+1}\\ &= \int_{-1}^{-1/(m-1)}\Delta_1(\zeta_1)\mathrm{d}\zeta_1 + \int_{1/(m-1)}^{1}\Delta_{n-m+1}(\zeta_{n-m+1})\mathrm{d}\zeta_{n-m+1}\end{aligned} \tag{36}$$

Let us assume that the integrated (and sampled) function $f(t)$ is a polynomial of the m-th order. Then coefficients s in equation (30) do not depend from i. Particularly:

$$s_{1,m} = s_{(n-m+1),m} \tag{37}$$

Then the sum of integrals in equation (36) is equal to 0 due to the symmetry of the limits. Finally:

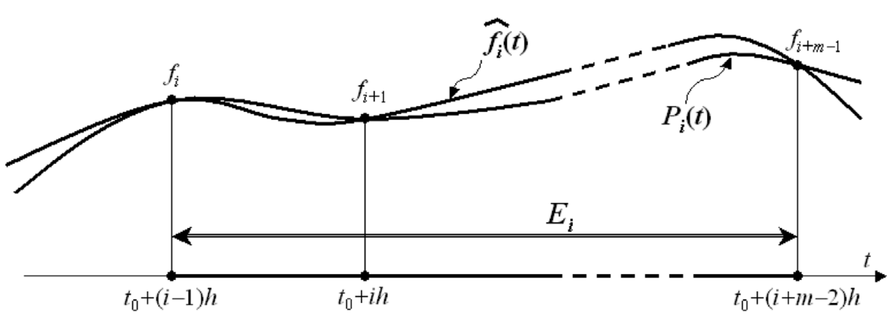

Figure 2.
Interpolation in the domain of finite element E_i

$$\int_{-1}^{1/(m-1)} \Delta_1(\zeta_1)\mathrm{d}\zeta_1 + \sum_{i=2}^{n-m} \int_{-1/(m-1)}^{1/(m-1)} \Delta_i(\zeta_i)\mathrm{d}\zeta_i + \int_{-1/(m-1)}^{1} \Delta_{n-m+1}(\zeta_{n-m+1})\mathrm{d}\zeta_{n-m+1} = 0 \tag{38}$$

Thus, the derived integration formula is exact when the integrated function is a polynomial of the m-th order. It can be noted again that this result is true when m is odd. If m is even, the function $\Delta_i(\zeta_i)$ is even and its integral is not equal to 0 though the limits are symmetric. Then the integration formula (equation (28)) is exact only when the integrated function is a polynomial of the $(m-1)$-st order. □

The derived integration rule (equation (28)) can be presented in the form of the following algorithm.

Algorithm. (0) Select the parameter m (the order of exactly integrated polynomial is shown in Table II). Define the time step h. Initialize Sum $= 0$ and $i = 0$. Allocate memory for a queue Q of m elements.

(1) Get a new function value f. Was the sampling successful, or was it the end of the process? If sampling was successful go to Step 2. Otherwise go to Step 5.

(2) Increment counter $i = i + 1$; save the current function value f in the queue Q. Saving f in a queue is understood as a standard procedure – shift queue's cells, forget the last cell of the queue and save f into the first cell of the queue.

(3) If $i > m$, then

$$\mathrm{Sum} = \mathrm{Sum} + f$$

Otherwise:

$$\mathrm{Sum} = \mathrm{Sum} + a(i)*f$$

(4) Go to Step 1.

(5) If $i \geq 2m$, then repeat for all j from 1 to m:

$$\mathrm{Sum} = \mathrm{Sum} + (a(j) - 1)*Q(j)$$

Otherwise display error message "Time series too short for the defined m" and STOP.

(6) Calculate the integral:

$$\mathrm{Sum} = h*\mathrm{Sum}$$

and STOP.

Working example

Holographic interferograms generated by a non-stationary motion of a micro cantilever beam are considered. It is assumed that the energy of the beam is concentrated in the first eigenmode only; the displacement of the end of the beam is explicitly defined at discrete time moments $t_0 + (i-1)h$, $i = 1, 2, \ldots, n$ (Figure 3). The sampling rate of the pulse laser is $h = 0{,}3\,\text{ms}$. Digital patterns of fringes must be reconstructed at different exposure times. It can be noted that the exposure times were selected without considering the divisibility of the number of sampling points what is a definite advantage over existing integration techniques. The number of sampling points at T_1 is $n_1 = 21$; at T_2 is $n_2 = 41$; at T_3 is $n_3 = 61$; at T_4 is $n_4 = 81$ (Figure 3).

In order to reconstruct the pattern of fringes in the hologram it is necessary to calculate the following integral (Vest, 1979):

$$\begin{aligned} L(x,y) &= \frac{1}{T^2}\left|\int_0^T \exp\left(j\frac{2\pi}{\lambda}a(x,y)f(t)\right)\mathrm{d}t\right|^2 \\ &= \left(\frac{1}{T}\int_0^T \cos\left(\frac{2\pi}{\lambda}\,a(x,y)f(t)\right)\mathrm{d}t\right)^2 \\ &\quad + \left(\frac{1}{T}\int_0^T \sin\left(\frac{2\pi}{\lambda}\,a(x,y)f(t)\right)\mathrm{d}t\right)^2 \end{aligned} \tag{39}$$

where x, y, co-ordinates in the plane of the hologram; L, intensity of illumination in the hologram; T, exposure time; j, imaginary unit; λ, laser wavelength; $a(x, y)$, eigenmode of the illuminated structure; $f(t)$, displacement from the state of equilibrium.

The order of the integration rule is selected $m = 9$. The procedure for generation of digital holographic interferograms is discussed in detail in Ragulskis *et al.* (2003). Numerically reconstructed patterns of fringes are presented in Figure 4. The patterns

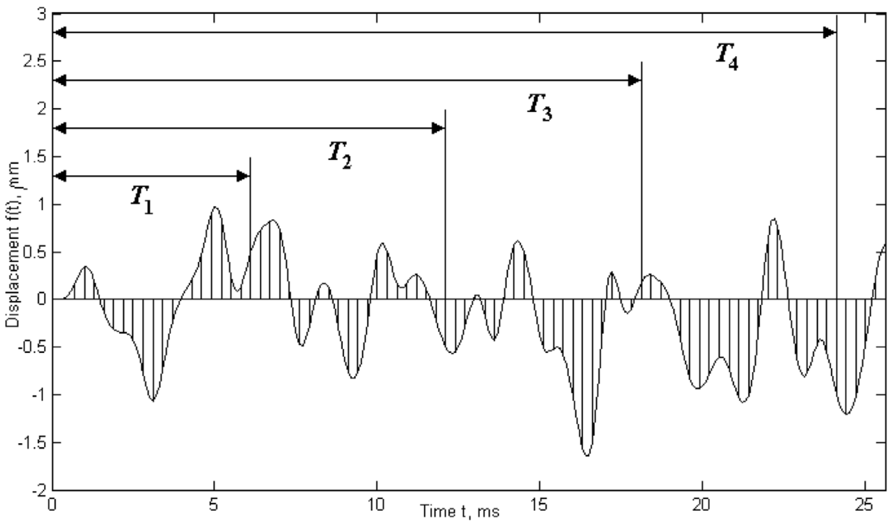

Figure 3.
Time history of the displacement of the cantilever beam

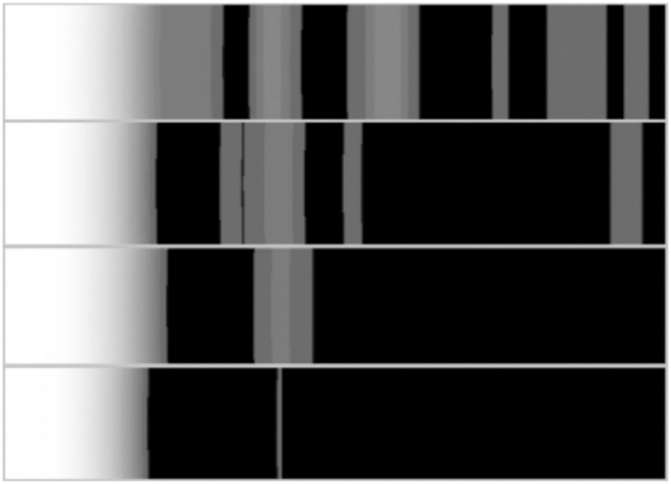

Figure 4.
Patterns of fringes at different exposure times

of fringes disappear for longer exposure times what is an indicator of stochastic vibrations Vest (1979). Finally, it can be noted that the proposed integration rule helped to construct realistic images with controlled accuracy and arbitrary selected exposure times.

Concluding remarks

The presented order adaptive integration rule has definite advantages over existing integration techniques. First of all, it does not require the divisibility of the number of sampling points of the integrand. This feature can be very useful in real time integration applications when the terminal moment of the sampling process is not predefined at the beginning of the experiment. Secondly, the developed integration technique has a feature of equivalently weighted internal nodes what makes its computational implementation very simple.

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