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The explosive divergence in iterative maps of matrices

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ABSTRACT

The effect of explosive divergence in generalized iterative maps of matrices is defined and described using formal algebraic techniques. It is shown that the effect of explosive divergence can be observed in an iterative map of square matrices of order 2 if and only if the matrix of initial conditions is a nilpotent matrix and the Lyapunov exponent of the corresponding scalar iterative map is greater than zero. Computational experiments with the logistic map and the circle map are used to illustrate the effect of explosive divergence occurring in iterative maps of matrices.

1. Introduction

Dynamical systems made of many units with global coupling are models of numerous important situations in physics and beyond, ranging from neural networks to hydrodynamics [1,2]. Discrete-time map-based models of networks have a long tradition in the physics of complex systems [3,4]. Coupled map lattices are spatially extended networks of dynamically evolving elements which exhibit a rich variety of dynamical and statistical phenomena. The phenomena of bifurcation behavior, multiple co-existing attractors and spatio-temporal intermittency in the context of coupled sine circle map and coupled logistic map lattices are studied in [5]. Collective behavior in coupled chaotic map lattices with random perturbations is discussed in [6]. Chaotic processes in coupled map lattices are estimated using symbolic vector dynamics in [7]. The synchronization threshold in coupled logistic map lattices is studied in [8]. Coupled map lattices are exploited in cryptographic applications in [9,10].

The main objective of this paper is to investigate not a coupled map lattice, but a single iterative map with a scalar discrete variable replaced by a square matrix of order 2. In other words, we do not extend the number of nodes in a network. In opposite, we extend the dimensionality of the problem in a single node. This is an alternative approach to studying complex systems. But the dynamics of the logistic map with a scalar discrete variable replaced by a square matrix of order 2 already becomes very complicated [11]. The scalar logistic map is one of the simplest and well-explored iterative maps. Nevertheless, such variable replacement in the logistic map introduces specific dynamical effects when the iterative process may diverge at certain eigenvalues of the matrix of initial conditions [11].

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Thus, before making any generalizations regarding the dimension of the square matrix of discrete variables, we aim to develop a theory describing the nonlinear dynamics of a general iterative map (not only the logistic map) with a scalar discrete variable replaced by a square matrix of order 2.

2. Preliminaries

Several properties of square matrices of order 2 will be discussed in this section. These properties are essential for studying properties of iterative maps of matrices.

2.1. Algebraic representation of matrices

Let us consider a square matrix of order 2:

$$X := \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \quad (1)$$

$x_{11}, \dots, x_{22} \in C$ and its eigenvalues $\lambda_1,\ \lambda_2 \in C$:

$$\lambda_{1,2} = \frac{1}{2}\left(TrX \pm \sqrt{dskX}\right), \quad (2)$$

where $TrX := x_{11} + x_{22};\ dskX := (x_{11} - x_{22})^2 + 4x_{12}x_{21}$.

Lemma 1. *Let eigenvalues of the matrix X are not equal*: $\lambda_1 \neq \lambda_2$. *Then it is possible to construct two idempotents* D_k:

$$D_k := \frac{1}{\lambda_k - \lambda_l}(X - \lambda_l I); \quad k, l = 1, 2; \quad k \neq l; \quad (3)$$

where I is the identity matrix; $I := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. *Matrices* D_k *satisfy following equalities:*

(i) $det\ D_k = 0$
(ii) $D_1 + D_2 = I$
(iii) $D_k \cdot D_l = \delta_{kl} D_k$; $k, l = 1,2$ where $\delta_{kl} := \begin{cases} 1, & k = l; \\ 0, & k \neq l. \end{cases}$

It is advantageous to introduce the definition of conjugate idempotents since $D_2 = I - D_1$ [11].

Lemma 2. *Let eigenvalues of the matrix X coincide:* $\lambda_1 = \lambda_2 = \lambda_0$. *Then the nilpotent N defined as:*

$$N := X - \lambda_0 I \quad (4)$$

satisfies following relationships:

(i) $N^2 = \Theta$
(ii) $det\ N = 0$

where $\Theta := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Nilpotents N and cN; $c \in C$ are similar nilpotents (N and Θ are similar nilpotents) [11].

Corollary 1. *If* $\lambda_1 \neq \lambda_2$ ($dsk\ X \neq 0$) *then the matrix X can be expressed as:*

$$X = \lambda_1 D_1 + \lambda_2 D_2. \quad (5)$$

If $\lambda_1 = \lambda_2 = \lambda_0$ *(dsk X = 0) then the matrix X can be expressed as:*

$$X = \lambda_0 I + N. \quad (6)$$

Definition 2.1. The matrix X is a type I matrix if it can be expressed in the form of Eq. (5) where $\lambda_1; \lambda_2$ are eigenvalues of X and $D_1; D_2$ are conjugate idempotents. The matrix X is a type II matrix if its eigenvalues are equal $\lambda_1 = \lambda_2 = \lambda_0$ and $N \neq \Theta$.

Let us notice that a scalar matrix $X = \lambda_0 I$ can be expressed in the form $\lambda_0 I = \lambda_0 D_1 + \lambda_0 D_2$ where $D_1; D_2$ is a pair of conjugate idempotents or in the form $\lambda_0 I = \lambda_0 I + N$ (where the nilpotent is Θ).

Lemma 3. *Let X_1' and X_1'' are two type I matrices and their idempotents are equal. Then idempotents of $X_1' \cdot X_1''$ and $X_1' + X_1''$ are also the same. Analogously, let X_2' and X_2'' are two type II matrices and their nilpotents are similar. Then the nilpotent of $X_2' \cdot X_2''$ and the nilpotent of $X_2' + X_2''$ is similar to the nilpotent of X_2' and the nilpotent of X_2''.*

Lemma 4. *Let X_1 be a type I matrix and X_2 be a type II matrix. Then powers X_1^n and X_2^n; $n = 0, 1, 2, \ldots$ read:*

$$X_1^n = \lambda_1^n D_1 + \lambda_2^n D_2; \tag{7}$$

$$X_2^n = \lambda_0^n I + n\lambda_0^{n-1} N; \tag{8}$$

where D_1 and D_2 are idempotents of X_1 and N is the nilpotent of X_2.

Proofs of Lemmas 1–4 are given in [11].

2.2. Parametric expressions of idempotents and nilpotents

It is advantageous to establish parametric expressions of idempotents D_1; D_2 and the nilpotent N.

Let us assume that X is a type I matrix. Then Eqs. (2) and (3) yield the parametric expressions of idempotents of X:

$$D_1 = \frac{1}{2}\begin{bmatrix} 1+\alpha & \beta \\ \frac{1-\alpha^2}{\beta} & 1-\alpha \end{bmatrix}; \quad D_2 = \frac{1}{2}\begin{bmatrix} 1-\alpha & -\beta \\ -\frac{1-\alpha^2}{\beta} & 1+\alpha \end{bmatrix}, \tag{9}$$

where $\alpha := \frac{x_{11}-x_{22}}{\sqrt{X}}$; $\beta := \frac{2x_{12}}{\sqrt{X}}$; $dskX = (\lambda_1 - \lambda_2)^2$.

Analogously, when X is a type II matrix, notations $\hat{\alpha} = x_{11} - x_{22}$; $\hat{\beta} := 2x_{12}$; Eq. (4) and the equality $dskX = 0$ yield:

$$N = \frac{1}{2}\begin{bmatrix} \hat{\alpha} & \hat{\beta} \\ -\frac{\hat{\alpha}^2}{\hat{\beta}} & -\hat{\alpha} \end{bmatrix}. \tag{10}$$

Thus, parametric expressions of idempotents D_1, D_2 and the nilpotent N exist for all values of parameters $\alpha, \hat{\alpha}, \beta, \hat{\beta} \in C$ except $\beta, \hat{\beta} = 0$.

3. The iterative map of matrices

The iterative map of matrices

$$X^{(n+1)} := f\left(X^{(n)}\right); \quad n = 0, 1, 2, \ldots \tag{11}$$

is considered in this paper, where $X^{(n)} = \begin{bmatrix} x_{11}^{(n)} & x_{12}^{(n)} \\ x_{21}^{(n)} & x_{22}^{(n)} \end{bmatrix}$; $x_{kl}^{(n)} \in R$; $k, l = 1, 2$ is a square matrix of order 2 and $f : R \to R$ is a scalar function.

Let us assume that the function $f(x)$ can be expanded into a series:

$$f(x) = \sum_{j=0}^{\infty} c_j \frac{x^j}{j!} \tag{12}$$

where $c_j \in R$; $j = 0, 1, \ldots$ and $x \in R$.

Theorem 1. *Let X be a type I matrix of order 2 and $f(x)$ can be expressed in the form (12). Then,*

$$f(X) = f(\lambda_1)D_1 + f(\lambda_2)D_2, \tag{13}$$

where λ_1, λ_2 are eigenvalues of X $(\lambda_1 \neq \lambda_2)$ and D_1, D_2 are idempotents of X.

Proof

$$f(X) = \sum_{j=0}^{+\infty} \frac{c_j}{j!}(\lambda_1 D_1 + \lambda_2 D_2)^j = \left(\sum_{j=0}^{+\infty} \frac{c_j}{j!}\lambda_1^j\right) D_1 + \left(\sum_{j=0}^{+\infty} \frac{c_j}{j!}\lambda_2^j\right) D_2 = f(\lambda_1)D_1 + f(\lambda_2)D_2. \qquad \square$$

Theorem 2. *Let X be a type II matrix of order 2 and $f(x)$ can be expressed in the form (12). Then,*

$$f(X) = f(\lambda_0)I + \dot{f}(\lambda_0)N, \tag{14}$$

where λ_0 is the recurrent eigenvalue ($\lambda_1 = \lambda_2 = \lambda_0$) and N is the nilpotent of X, $\dot{f}(\lambda_0)$ denotes the derivative of f with respect to x at λ_0.

Proof

$$f(X) = \sum_{j=0}^{+\infty} \frac{c_j}{j!} (\lambda_0 I + N)^j = \sum_{j=0}^{+\infty} \frac{c_j}{j!} \left(\lambda_0^j I + j\lambda_0^{j-1} N \right) = f(\lambda_0)I + \dot{f}(\lambda_0)N. \qquad \square$$

Corollary 2. Let the iterative map of matrices reads:

$$X^{(n+1)} := f\left(X^{(n)}\right) = \sum_{j=0}^{\infty} c_j \frac{\left(X^{(n)}\right)^j}{j!} \quad n = 0, 1, 2, \ldots \tag{15}$$

Let the matrix of initial conditions be a type I matrix: $X^{(0)} = \lambda_1 D_1 + \lambda_2 D_2$. Then, Corollary 3.1 yields the following iterative relationship: $\lambda_1^{(n+1)} D_1 + \lambda_2^{(n+1)} D_2 = f(\lambda_1^{(n)}) D_1 + f(\lambda_2^{(n)}) D_2$; $n = 0, 1, 2, \ldots$

Comment 1. Corollary 2 yields a straightforward iterative relationship describing the evolution of eigenvalues of the type I matrix:

$$\begin{cases} \lambda_1^{(n+1)} = f\left(\lambda_1^{(n)}\right); \\ \lambda_2^{(n+1)} = f\left(\lambda_2^{(n)}\right); \end{cases} \quad n = 0, 1, 2, \ldots \tag{16}$$

In other words, matrices generated by the iterative map preserve the same idempotents D_1, D_2 if the matrix of initial conditions is a type I matrix.

Corollary 3. *Let the iterative map of matrices reads: $X^{(n+1)} := f(X^{(n)})$; $n = 0, 1, 2, \ldots$ Let the matrix of initial conditions be a type II matrix: $X^{(0)} = \lambda_0^{(0)} I + \mu_0^{(0)} N$; $\mu_0^{(0)} = 1$. Then, $\lambda_0^{(n+1)} I + \mu_0^{(n+1)} N = f\left(\lambda_0^{(n)}\right) I + \mu_0^{(n)} \dot{f}\left(\lambda_0^{(n)}\right) N$; $n = 0, 1, 2, \ldots$*

Comment 2. Corollary 3 yields a straightforward iterative relationship describing the evolution of the eigenvalue of the type II matrix:

$$\begin{cases} \lambda_0^{(n+1)} = f\left(\lambda_0^{(n)}\right); \\ \mu_0^{(n+1)} = \mu_0^{(n)} \dot{f}\left(\lambda_0^{(n)}\right); \end{cases} \mu_0^{(n)} \in R; \quad n = 0, 1, 2, \ldots. \tag{17}$$

In other words, the iterative map generates a sequence of type II matrices (if only the matrix of initial conditions is a type II matrix). The evolution of the supplementary variable $\mu_0^{(n+1)}$ can be rewritten in the following form:

$$\mu_0^{(n+1)} = \prod_{k=0}^{n} \dot{f}\left(\lambda_0^{(k)}\right); \quad \mu_0^{(0)} = 1; \quad n = 0, 1, 2, \ldots. \tag{18}$$

Definition 3.3. The explosive divergence occurs in the iterative map of matrices if

$$\lim_{n \to +\infty} \prod_{k=0}^{n} |\dot{f}\left(\lambda_0^{(k)}\right)| = +\infty, \tag{19}$$

and the eigenvalue of the iterative type II matrix remains bounded; $|\lambda_0^{(n)}| \le M < +\infty$; $n = 0, 1, 2, \ldots$.

Comment 3. Definition 3.3 implies that the effect of explosive divergence in the iterative map of matrices cannot be observed if the matrix of initial conditions is a type I matrix.

Lyapunov exponent of a scalar one-dimensional iterative map reads [12]:

$$\tilde{\lambda} = \frac{1}{n} \sum_{j=0}^{n-1} \ln |\dot{f}\left(\lambda_0^{(j)}\right)|, \tag{20}$$

where $\tilde{\lambda}$ is a numerical estimate of the Lyapunov exponent. Lyapunov exponents are calculated for a sequence of iterative values $\lambda_0^{(j)}$ after all transient processes have ceased down.

Corollary 4. *The explosive divergence occurs in an iterative map of matrices if the matrix of initial conditions is a type II matrix and the Lyapunov exponent of the corresponding scalar iterative map is greater than zero.*

Proof. If the matrix of initial conditions is a type II matrix then the evolution of its eigenvalue λ_0 is governed by the first equality in Eq. (17). Eq. (18) implies that the derivative of the map function is evaluated starting from $\lambda_0^{(0)}$. But the necessary condition for Eq. (18) to hold true (as n tends to infinity) implies that the following condition must hold true:

$$\lim_{p\to\infty}\prod_{k=m}^{m+p}|\dot{f}\left(\lambda_0^{(k)}\right)| > 1 \tag{21}$$

where m represents the length of transient processes. But the Lyapunov exponent of a stable periodic attractor is negative; Eq. (21) and Eq. (19) do not hold true then. Therefore the effect of explosive divergence cannot be observed if the scalar iterative map $\lambda_0^{(n+1)} = f\left(\lambda_0^{(n)}\right)$ evolves into a stable period regime. □

It can be noted that the numerical value of the parameter $\mu_0^{(n)}$ can be considered as the estimate of the system stability.

4. Computational experiments

It has been shown in the previous section that the effect of explosive divergence occurs in the iterative map of matrices if the matrix of initial conditions is a type II matrix and the Lyapunov exponent of the corresponding scalar iterative map is greater than zero. We will perform computational experiments with the logistic map and the circle map to illustrate these theoretical results. The logistic map is a paradigmatic model used to illustrate how complex behavior can arise from very simple non-linear dynamical equations [13,14]:

$$y^{(n+1)} = ay^{(n)}\left(1 - y^{(n)}\right); \tag{22}$$

where n is the iteration number; $n = 0, 1, 2, \ldots$; $a \in \mathrm{R}$ is the parameter of the logistic map and $y^{(0)}$ is the initial condition. The logistic map is also successfully exploited to illustrate the concept of H-rank at the onset to chaos [15]. The circle map is a paradigmatic model used to illustrate the effect of phase locking and to study the dynamical behavior of a beating heart [16,17]:

$$\theta^{(n+1)} = \theta^{(n)} + \Omega - \frac{K}{2\pi}sin\left(2\pi\theta^{(n)}\right); \tag{23}$$

where n is the iteration number; is a normalized polar angle in the interval $[0; 1]$; K is the coupling strength and is the driving phase.

Fig. 1 illustrates the evolution of the circle map of matrices when the matrix of initial conditions is a type I matrix. It is clear that the effect of divergence cannot be observed in such a system; both eigenvalues of the iterative matrix are locked in the 3:7 mode. The effect of divergence cannot be observed in the evolution of the logistic map of matrices when the matrix of initial conditions is a type I matrix even though the dynamics of the corresponding scalar map is chaotic (Fig. 2; the Lyapunov exponent of the scalar iterative map is equal to 0.4312).

Fig. 3 shows the evolution of the logistic map of matrices when the matrix of initial conditions is a type II matrix. The initial tendency to diverge can be observed until the transient processes do not cease down. The Lyapunov exponent of the iterative map

$$\lambda_0^{(n+1)} = a\lambda_0^{(n)}\left(1 - \lambda_0^{(n)}\right); \tag{24}$$

is equal to $-0.8723 < 0$ (at $a = 3.5$). Thus, in the long run, the logistic map of matrices quiets down.

Fig. 4 illustrates the effect of explosive divergence in the logistic map of matrices. Now, the Lyapuvov exponent of Eq. (24) is equal to $0.4312 > 0$ (at $a = 3.8$) and the system experiences a violent divergence (computations are terminated due to the numerical overflow). Analogously, the quieting of the circle map of matrices is illustrated in Fig. 5; the Lyapunov exponent of the iterative map

$$\lambda_0^{(n+1)} = \lambda_0^{(n)} + \Omega - \frac{K}{2\pi}sin\left(2\pi\lambda_0^{(n)}\right) \tag{25}$$

is equal to $-0.0546 < 0$ (at $K = 4.4$ and $\Omega = 0.428$). On the other hand, the effect of explosive divergence is observed in the circle map of matrices at $K = 4.6$ and $\Omega = 0.428$ (Fig. 6); the Lyapunov exponent of the scalar iterative map Eq. (25) is equal to $0.4208 > 0$ then.

Two-dimensional discrete chaotic maps are used in a variety of applications. Coupled deterministic models based on the circle map and the two-dimensional standard map is used to simulate electrocardiographic signals in [18]. The extended two-dimensional chaotic cat map is exploited for the development of new digital watermarking algorithm in [19]. The generalized two-dimensional Baker map is used to solve the XOR problem in [20] and to develop new symmetric block encryption schemes in [21]. A modified two-dimensional Henon map is exploited to generate multi-fold strange attractors via period-doubling bifurcation routes to chaos in [22]. A finite-state two-dimensional torus map is used as a diffusion layer in a chaotic public-key encryption algorithm in [21].

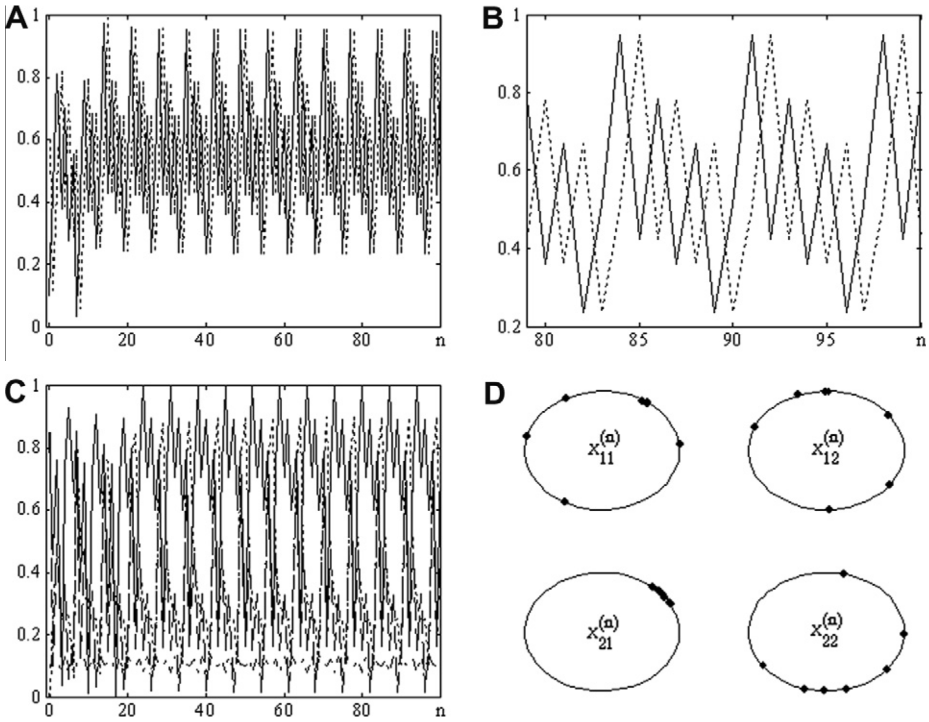

Fig. 1. The circle map of matrices does not exhibit the effect of explosive divergence when the matrix of initial conditions is a type I matrix ($\alpha = 2, \beta = 8$, $\lambda_1^{(0)} = 0.1$, $\lambda_2^{(0)} = 0.6$); parameters K = 0.96 and Ω = 0.428 result into 3:7 synchronization. A shows the evolution of eigenvalues in the interval $0 \leqslant n \leqslant 100$ (n is the iteration number); B is the zoomed image of A in the interval $80 \leqslant n \leqslant 100$. C shows the evolution of $x_{11}^{(n)}, x_{12}^{(n)}, x_{21}^{(n)}, x_{22}^{(n)}$ and D shows phases of each element of the iterative matrix after transient processes have ceased down.

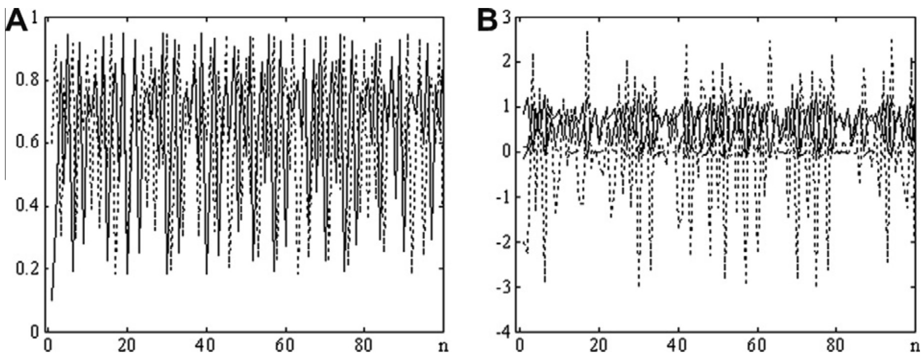

Fig. 2. The logistic map of matrices does not exhibit the effect of explosive divergence when the matrix of initial conditions is a type I matrix ($\alpha = 2$, $\beta = 8$, $\lambda_1^{(0)} = 0.1$, $\lambda_2^{(0)} = 0.6$); parameter a = 3.8. A shows the evolution of eigenvalues in the interval $0 \leqslant n \leqslant 100$ (n is the iteration number); B shows the evolution of $x_{11}^{(n)}, x_{12}^{(n)}, x_{21}^{(n)}, x_{22}^{(n)}$.

We propose a novel application of the iterative map of matrices for the construction of generalized two-dimensional discrete maps. Let us consider a vector $(p^{(n)}, q^{(n)})$; $p^{(n)}, q^{(n)} \in R; n = 0, 1, 2, \ldots$. Then, the generalized two-dimensional iterative discrete map reads:

$$\left(p^{(n+1)}, q^{(n+1)}\right) := \left(p^{(n)}, q^{(n)}\right) \cdot X^{(n)}; \quad X^{(n+1)} = F\left(X^{(n)}\right); \quad n = 0, 1, 2, \ldots. \tag{26}$$

where $(p^{(0)}, q^{(0)}) \neq (0, 0)$; $X^{(0)}$ is a square matrix of order 2. Such an iterative compound map of matrices generalizes two-dimensional discrete maps. Moreover the dynamics of such an iterative map is more rich compared to standard discrete maps. Particularly the effect of explosive divergence can be observed in (25) if the matrix of initial conditions $X^{(0)}$ is a type II matrix and the Lyapunov exponent of the scalar map $\lambda_0^{(n+1)} = F(\lambda_0^{(n)})$, $n = 0, 1, 2, \ldots$ is positive.

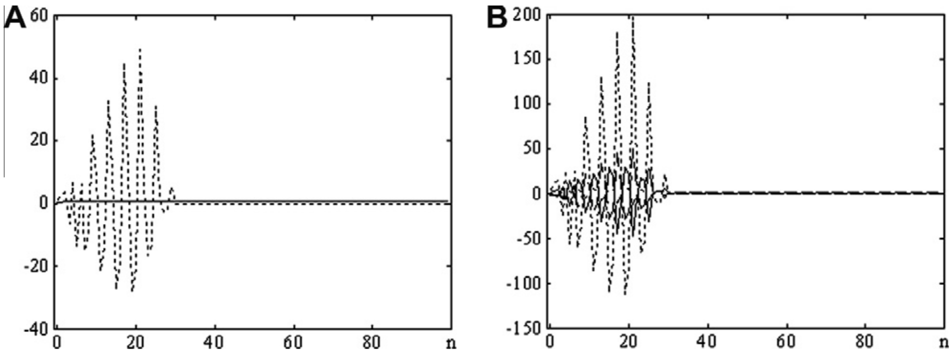

Fig. 3. The logistic map of matrices shows an initial tendency to diverge but quiets down when transient processes cease down. The matrix of initial conditions is a type II matrix ($\alpha = 2$, $\beta = 8$, $\lambda_0^{(0)} = 0.1$); $a = 3.5$. A shows the evolution of the eigenvalue (represented by a solid line) and the parameter $\mu_0^{(n)}$ (a dashed line). B shows the evolution of $x_{11}^{(n)}, x_{12}^{(n)}, x_{21}^{(n)}, x_{22}^{(n)}$.

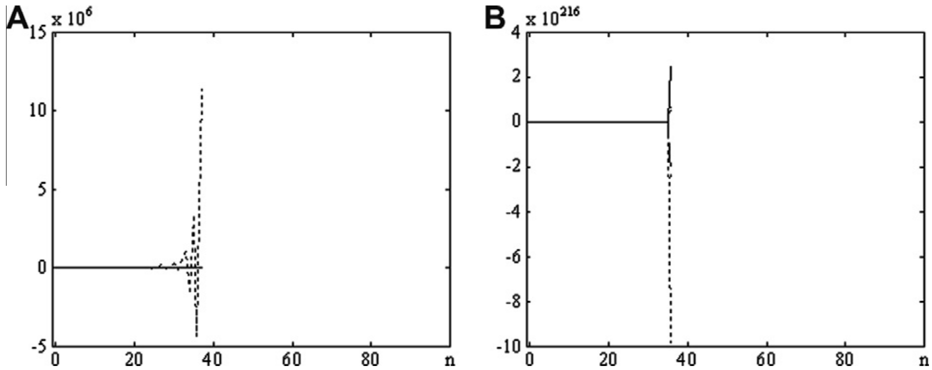

Fig. 4. The logistic map of matrices exhibits the effect of explosive divergence. The matrix of initial conditions is a type II matrix ($\alpha = 2$, $\beta = 8$, $\lambda_0^{(0)} = 0.1$); $a = 3.8$. A shows the evolution of the eigenvalue (represented by a solid line) and the parameter $\mu_0^{(n)}$ (a dashed line). B shows the evolution of $x_{11}^{(n)}, x_{12}^{(n)}, x_{21}^{(n)}, x_{22}^{(n)}$.

Let $X^{(n)}$ is a nilpotent operator:

$$X^{(n)} = \lambda_0^{(n)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \mu_0^{(n)} \frac{1}{2} \begin{bmatrix} \hat{\alpha} & \hat{\beta} \\ -\frac{\hat{\alpha}^2}{\hat{\beta}} & -\hat{\alpha} \end{bmatrix}. \quad (27)$$

Then

$$\begin{aligned} \left(p^{(n+1)}, q^{(n+1)}\right) &= \left(p^{(n)}, q^{(n)}\right) X^{(n)} = \left(\left(\lambda_0^{(n)} + \frac{\mu_0^{(n)} \hat{\alpha}}{2} \right) p^{(n)} - \frac{\mu_0^{(n)} \hat{\alpha}^2}{2\hat{\beta}} q^{(n)}, \frac{\mu_0^{(n)} \hat{\beta}}{2} p^{(n)} + \left(\lambda_0^{(n)} - \frac{\mu_0^{(n)} \hat{\alpha}}{2} \right) q^{(n)} \right) \\ &= \left(\lambda_0^{(n)} p^{(n)}, \lambda_0^{(n)} q^{(n)} \right) + \frac{\mu_0^{(n)}}{2} \left(\hat{\alpha} p^{(n)} - \frac{\hat{\alpha}^2}{\hat{\beta}} q^{(n)}, \hat{\beta} p^{(n)} - \hat{\alpha} q^{(n)} \right) \end{aligned} \quad (28)$$

Thus

$$\begin{cases} p^{(n+1)} = \lambda_0^{(n)} p^{(n)} + \frac{\mu_0^{(n)}}{2} \left(\hat{\alpha} p^{(n)} - \frac{\hat{\alpha}^2}{\hat{\beta}} q^{(n)} \right) \\ q^{(n+1)} = \lambda_0^{(n)} q^{(n)} + \frac{\mu_0^{(n)}}{2} \left(\hat{\beta} p^{(n)} - \hat{\alpha} q^{(n)} \right) \end{cases}. \quad (29)$$

It can be noted that parameters $\hat{\alpha}$ and $\hat{\beta}$ are coordinates of eigenvectors and $\lambda_0^{(n)}$ is the eigenvalue of the nilpotent operator $X_0^{(n)}$:

$$\left(\hat{\alpha}, \hat{\beta}\right) \xrightarrow{X^{(n)}} \lambda_0^{(n)} \left(\hat{\alpha}, \hat{\beta}\right). \quad (30)$$

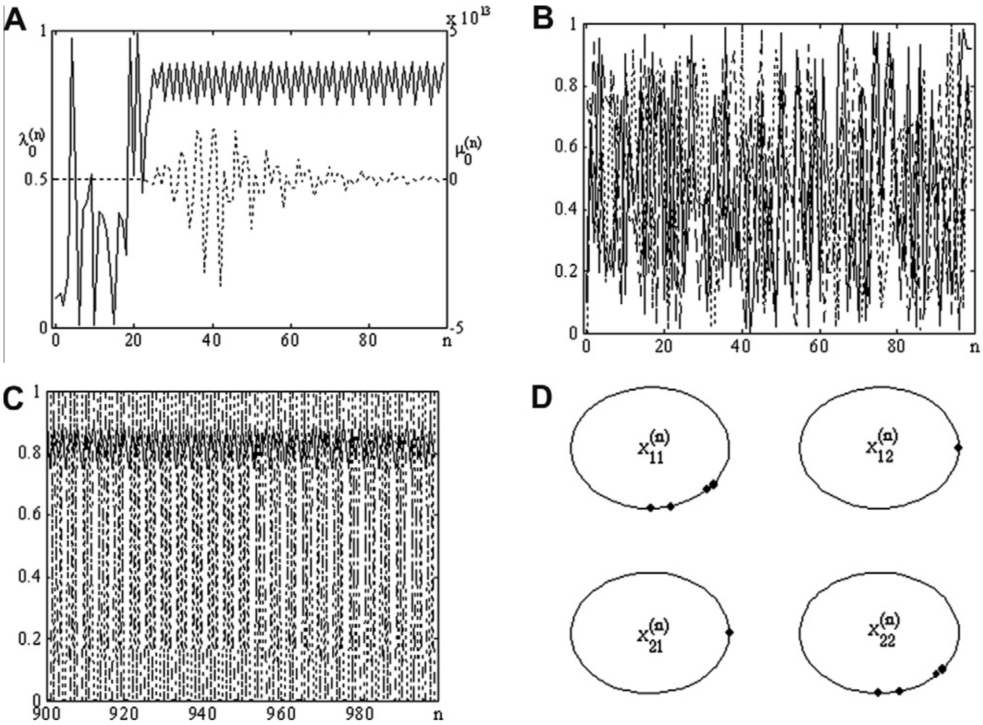

Fig. 5. The circle map of matrices shows an initial tendency to diverge but quiets down when transient processes cease down. The matrix of initial conditions is a type II matrix ($\alpha = 2$, $\beta = 8$, $\lambda_0^{(0)} = 0.1$); $K = 4.4$ and $\Omega = 0.428$. A shows the evolution of the eigenvalue (a solid line) and the parameter $\mu_0^{(n)}$ (a dashed line). B shows the evolution of $x_{11}^{(n)}, x_{12}^{(n)}, x_{21}^{(n)}, x_{22}^{(n)}$. C and D illustrate the evolution of the system after transient processes have ceased down (elements of the iterative matrix are shown in C and phases of elements of the iterative matrix are shown in D).

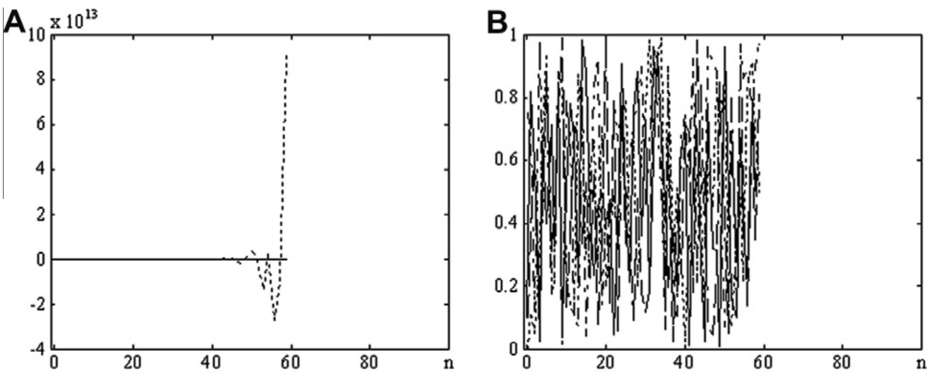

Fig. 6. *The circle map of matrices exhibits the effect of explosive divergence. The matrix of initial conditions is a type II matrix (*$\alpha = 2$, $\beta = 8$, $\lambda_0^{(0)} = 0.1$*);* $K = 4.6$ and $\Omega = 0.428$. A shows the evolution of $x_{11}^{(n)}, x_{12}^{(n)}, x_{21}^{(n)}, x_{22}^{(n)}$; B shows the evolution of the eigenvalue (a solid line) and the parameter $\mu_0^{(n)}$ (a dashed line).

5. Concluding remarks

The effect of explosive divergence in generalized iterative maps of matrices is described in this paper. Necessary and sufficient conditions for the existence of such divergence are derived and illustrated by computational experiments. So far, we have investigated maps of square matrices of order 2 only. Iterative maps of higher order and concrete applications where the effect of explosive divergence can be exploited as a factor ensuring the additional security of the encoding scheme, for example, are definite objects of future research.

Acknowledgements

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