Chaos, Solitons and Fractals 104 (2017) 625–634

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

Operator-based approach for the construction of analytical soliton solutions to nonlinear fractional-order differential equations

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ARTICLE INFO

Article history:
Received 17 August 2017
Accepted 18 September 2017

2010 MSC:
34A08
34A25
26A33
35C05

Keywords:
Fractional differential equation
Operator calculus
Analytical solution
Closed-form solution
Soliton solution

ABSTRACT

An operator-based framework for the construction of analytical soliton solutions to fractional differential equations is presented in this paper. Fractional differential equations are mapped from Caputo algebra to Riemann-Liouville algebra in order to preserve the additivity of base function powers under multiplication. The proposed technique is used for the construction of solutions to a class of fractional Riccati equations. Recurrence relations between power series parameters yield generating functions which are used to construct explicit expressions of closed-form solutions.

1. Introduction

Even though the idea of fractional differentiation dates back to Leibniz's letter to L'Hopital at the end of the 17th century [1], topics related to fractional derivatives and fractional differential equations have recently become the focus of vigorous research. The reason for renewed attention to this field has been due to the discovery of applications of fractional differential equations in many areas of science and engineering [2,3]. A comprehensive review on the application of fractional calculus in physics can be found in [4]. Numerical methods for fractional differential equations are considered in [5]. Examples of practical applications of fractional calculus are given in [6].

Fractional differential equations play a central role in viscoelasticity research, which was prompted by the wide use of polymer materials in engineering applications [7]. Fractional derivatives enable the modeling of materials that have memory of their previous deformations [8,9]. It is shown in [10] that models based on fractional derivative can be used to approximate the behavior of materials in Magnetic Resonance Elastography. A model of viscoelasticity employing fractional order derivatives is incorporated into Maxwell's equations in [11]. Fractional order controllers of a hexapod robot with viscous friction leg joints is considered in [12]. A particle-tracking approach using fractional differential equations is applied to simulate fractional diffusion-reaction processes in [13].

The introduction of fractional derivatives to the harmonic oscillator has led to new oscillatory phenomena [14]. An interpretation of fractional oscillators as an ensemble average of ordinary harmonic oscillators governed by a stochastic time arrow is given in [15]. The frequency and amplitude of the fractional-order Duffing oscillator are evaluated using a describing function method in [16]. The feedback control system is used as an active control strategy to control horseshoes chaos in a driven Rayleigh oscillator with fractional deflection in [17]. The eigenvalue spectrum of a fractional quantum harmonic oscillator is numerically investigated in [18].

Fractional derivatives are one of the novel concepts in recent advances of biomedical research [19]. In [20], the order of a fractional derivative is used to model two-stage human memory phenomena. It is demonstrated in [21] that nonlinear fractional-order models can be used to study valproic acid pharmacokinetics. The effect of fractional derivative order on a delayed predator-prey system with harvesting terms is considered in [22]. Lung parenchyma viscoelasticity is studied using fractional-order models in [23]. A

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https://doi.org/10.1016/j.chaos.2017.09.026

fractional calculus approach is used to describe the dynamic behavior of cartilage in [24].

A number of new techniques for the construction of solutions to fractional differential equations have recently been developed. The waveform relaxation method is used to solve fractional differential-algebraic equations that arise in integrated circuits with new memory materials [25]. A spectral decomposition is used in conjunction with Fourier and Laplace transforms to solve the time-fractional diffusion equation in [26]. Artificial neural networks are used to construct numerical solutions to a number of different fractional differential equations in [27]. Blow-up solutions to the nonlinear fractional Schrödinger equation are studied using variational arguments and profile decomposition theory in [28].

Solutions to fractional wave-diffusion equations, modified anomalous fractional sub-diffusion equations and time-fractional telegraph equations are constructed using semi-analytical techniques based on the Fourier series expansion in [29]. Laplace and Fourier transforms with respect to both time and space variables are used to obtain fundamental solutions to the Riesz fractional advection-dispersion equation in [30]. Operational calculus of the Misukiński type is used to solve initial value problems on fractional differential equations with generalized Riemann-Liouville derivatives in [31]. Three time-splitting schemes for nonlinear time-fractional differential equations with smooth solutions are proposed in [32]. Second order implicit-explicit time-stepping schemes for nonlinear fractional differential equations with nonsmooth solutions are considered in [33]. The fractional-order Legendre operational matrix is used to construct approximate solutions to the fractional Riccati equation in [34].

A number of novel definitions of fractional derivatives have recently been introduced. Generalized fractional operators are applied to a particular class of nonstandard Lagrangians in [35]. A mixed integro-differential operator of the Erdélyi-Kober type that generalizes Riemann-Liouville and Caputo derivatives is introduced in [36]. Fractional calculus of variations based on the extended Erdélyi-Kober fractional integral is constructed in [37].

The main objective of this paper is to present an operator-based framework for the construction of closed-form analytical solutions to fractional differential equations. To better illustrate this new approach, we analyse one of the simplest nonlinear fractional-order models – the Riccati equation:

$$\left(\sqrt{x}\right)^k \left({}^C\mathbf{D}^{1/2}\right)^k y = B_0 + B_1 y + B_2 y^2; \quad k = 1, 2, \dots, \tag{1}$$

where ${}^C\mathbf{D}^{1/2}$ denotes the Caputo fractional derivative of order $\frac{1}{2}$; $B_0, B_1, B_2 \in \mathbb{R}$.

The Caputo fractional derivative has been selected for this study because of its wide range of applications. Futhermore, the selection of a classical fractional derivative enables to more clearly illustrate the presented technique for the construction of closed-form solutions to fractional differential equations.

To aid clarity of exposition of the presented method for the construction of analytical solutions to (1), only fractional derivatives of order $\frac{1}{2}$ are considered. Derivations and computations given in this paper can be generalized for fractional derivatives of rational order $\frac{m}{n}$.

2. Motivation for (1)

Let us consider the ordinary Riccati equation with constant coefficients:

$$\frac{dz}{dx} = B_0 + B_1 z + B_2 z^2, \quad z(x_0) = z_0; \quad x_0, z_0, B_0, B_1, B_2 \in \mathbb{R}. \tag{2}$$

It is well-known that all solutions to (2) are kink solitary solutions [38]:

$$z = z_2 \frac{\exp\left(B_2(z_1 - z_2)(x - x_0)\right) - \frac{z_1(z_0 - z_2)}{z_2(z_0 - z_1)}}{\exp\left(B_2(z_1 - z_2)(x - x_0)\right) - \frac{z_0 - z_2}{z_0 - z_1}}, \tag{3}$$

where $z_1, z_2 \in \mathbb{C}$ are roots of the polynomial $B_0 + B_1 z + B_2 z^2$.

Direct solution of (2) is not straightforward for many of analytical solution construction methods. To counter this a typical independent variable substitution is used:

$$t = \exp(\eta x), \quad z(x) = \widehat{z}(t); \tag{4}$$

This transforms (5) into the Riccati equation with variable coefficients:

$$\eta t \frac{d\widehat{z}}{dt} = B_0 + B_1 \widehat{z} + B_2 \widehat{z}^2. \tag{5}$$

The general solution to (5) is a meromorphic function. It can be directly constructed using operator methods [39,40]:

$$\widehat{z} = \frac{\alpha_1 t - \beta_1}{\alpha_0 t - \beta_0}, \quad \alpha_0, \alpha_1, \beta_0, \beta_1 \in \mathbb{R}. \tag{6}$$

Thus equation (1) can be considered as a generalization of (5) in respect of Caputo fractional derivatives.

3. Preliminaries: main concepts and definitions

In this section, main concepts and definitions of the approach used to construct solutions to fractional-order nonlinear differential equations are given.

3.1. Power series extension

Functions that are analyzed in this paper are represented by the following power series:

$$y(z) = \sum_{j=0}^{+\infty} a_j \frac{z^j}{j!}, \quad z, a_j \in \mathbb{C}. \tag{7}$$

Coefficients a_j are constructed using operator techniques. The following two cases with respect to convergence of (7) in the Cauchy sense are considered:

1. Series (7) converges for $|z| < R$; $R > 0$. Then (7) can be extended to a wider complex domain (that does not include the singularities of (7)) using classical extension techniques. Letting $x \in \mathbb{R}$ be the argument of this extended function yields a real-argument power series $y(x)$ that is defined for values of x not necessarily in the convergence radius $|x| < R$. In this case, the extended function $y(x)$ and its power series representation are considered congruent.
2. Series (7) converges only for $|z| = 0$. In this case, the series is divergent in the Cauchy sense. However, such series still do contain important information [41]. Thus (7) can be considered as structural solutions to fractional differential equations without seeking congruent extended functions.

3.2. Fractional power series

For any $x \geq 0$, the following base functions are defined:

$$z_j = z_j(x) := \frac{x^{\frac{j-1}{2}}}{\Gamma\left(\frac{j+1}{2}\right)}, \quad j = 0, 1, \dots; \tag{8}$$

where Γ is the gamma function:

$$\Gamma(x) = \int_0^{+\infty} \xi^{x-1} \exp(-\xi)\, d\xi. \tag{9}$$

Note that Γ satisfies the following well-known equalities [42]:

$$\Gamma(1+x) = x\Gamma(x), \quad x > -1; \tag{10}$$

$$\Gamma(n+1) = n!, \quad n = 0, 1, \ldots; \tag{11}$$

$$\Gamma(x) = \pm\infty, \quad x = 0, -1, \ldots; \tag{12}$$

$$\Gamma\left(\frac{n}{2}\right) = \begin{cases} \frac{(n-2)!!\sqrt{\pi}}{2^{\frac{n-1}{2}}}, & n = 1, 3, \ldots; \\ \frac{(n-2)!!\sqrt{2}}{2^{\frac{n-1}{2}}}, & n = 2, 4, \ldots \end{cases}, \quad (-1)!! = 0!! := 1. \tag{13}$$

Fractional power series $\widehat{\omega}(x)$ are constructed using base functions (8):

$$\widehat{\omega}(x) := \sum_{j=0}^{+\infty} c_j z_j, \quad c_j \in \mathbb{C}. \tag{14}$$

Series (14) can be rewritten as:

$$\widehat{\omega}(x) = \omega_{-1}(x) + \omega_0(x) + \sqrt{x}\omega_1(x), \tag{15}$$

where

$$\omega_{-1}(x) = \frac{c_0}{\sqrt{\pi x}}; \tag{16}$$

$$\omega_0(x) = \sum_{k=0}^{+\infty} c_{2k+1} \frac{x^k}{k!}; \tag{17}$$

$$\omega_1(x) = \sum_{l=1}^{+\infty} \frac{l!\, c_{2l}}{\Gamma\left(l + \frac{1}{2}\right)} \frac{x^l}{l!}. \tag{18}$$

3.3. Linear spaces of fractional power series

In addition to the well-known Leibniz series:

$$\mathbb{F}_0 := \left\{ \sum_{j=0}^{+\infty} a_j \frac{x^j}{j!} \;\middle|\; a_j \in \mathbb{C} \right\}, \tag{19}$$

the following fractional power series are used in this paper:

$$^C\mathbb{F} := \left\{ \sum_{j=1}^{+\infty} c_j z_j \;\middle|\; c_j \in \mathbb{C} \right\}; \tag{20}$$

$$\mathbb{F} := \left\{ \sum_{j=0}^{+\infty} c_j z_j \;\middle|\; c_j \in \mathbb{C} \right\}, \tag{21}$$

Elements of sets (20), (21) are called Caputo and Riemann-Liouville series respectively.

Sets (19)–(21) form linear spaces over the complex field under standard sum and product by a scalar operations. Note that $\omega_{-1} = \omega_1 = 0$ for Leibniz series; $\omega_{-1} = 0$ for Caputo series and $\omega_{-1}, \omega_0, \omega_1 \neq 0$ for Riemann-Liouville series. Thus:

$$\mathbb{F}_0 \subset {}^C\mathbb{F} \subset \mathbb{F}. \tag{22}$$

Furthermore, the following relations hold true:

$$^C\mathbb{F} = \mathbb{F}_0 \oplus \sqrt{x}\mathbb{F}_0; \tag{23}$$

$$\mathbb{F} = {}^C\mathbb{F} \oplus \{\omega_{-1}(x)\}, \tag{24}$$

where $\oplus$ denotes the direct sum operation.

3.4. Fractional differentiation and integration operators in Riemann-Liouville linear space

Definition 3.1. Linear operators $\mathbf{I}^{1/2}$, $\mathbf{D}^{1/2}$, defined by the following relations [43]:

$$\mathbf{I}^{1/2} x^{\frac{j-1}{2}} := \frac{\Gamma\left(\frac{j+1}{2}\right)}{\Gamma\left(\frac{j}{2}+1\right)} x^{\frac{j}{2}}, \quad j = 0, 1, \ldots; \tag{25}$$

$$\mathbf{D}^{1/2} x^{\frac{j-1}{2}} := \frac{\Gamma\left(\frac{j+1}{2}\right)}{\Gamma\left(\frac{j}{2}\right)} x^{\frac{j}{2}-1}, \quad j = 0, 1, \ldots, \tag{26}$$

are called Riemann-Liouville integration and differentiation operators respectively.

Note that the fractional derivative of the leading term for Riemann-Liouville series is zero:

$$\mathbf{D}^{1/2} x^{-\frac{1}{2}} = \frac{\sqrt{\pi}}{\Gamma(0)} x^{-1} = \frac{\sqrt{\pi}}{\pm\infty} x^{-1} = 0, \tag{27}$$

however the derivative of a constant $c \in \mathbb{C}$ reads:

$$\mathbf{D}^{1/2} c = c\mathbf{D}^{1/2} 1 = c z_0 = \frac{c}{\sqrt{\pi x}}. \tag{28}$$

Furthermore, (25) and (26) together with (8) yield:

$$\mathbf{I}^{1/2} z_j = z_{j+1}, \quad j = 0, 1, \ldots; \tag{29}$$

$$\mathbf{D}^{1/2} z_j = \begin{cases} 0, & j = 0; \\ z_{j-1}, & j = 1, 2, \ldots. \end{cases} \tag{30}$$

Let $\widehat{f} \in \mathbb{F}$. Then (29) and (30) yield:

$$\mathbf{I}^{1/2} \widehat{f} = \sum_{j=0}^{+\infty} c_j z_{j+1} = \sum_{j=1}^{+\infty} c_{j-1} z_j; \tag{31}$$

$$\mathbf{D}^{1/2} \widehat{f} = \sum_{j=1}^{+\infty} c_j z_{j-1} = \sum_{j=0}^{+\infty} c_{j+1} z_j. \tag{32}$$

Corollary 3.1. *It can be observed that if $f \in {}^C\mathbb{F}$ then (22) and (32) yield that $\mathbf{D}^{1/2} f \in \mathbb{F}$. The converse statement is also true: if $\widehat{f} \in \mathbb{F}$, then (31) yields that $\mathbf{I}^{1/2}\widehat{f} \in {}^C\mathbb{F}$.*

Corollary 3.2. *Let $f \in {}^C\mathbb{F}$, $\widehat{f} \in \mathbb{F}$. Then, (25) and (26) yield:*

$$\mathbf{D}^{1/2}\mathbf{I}^{1/2} \widehat{f} = \widehat{f}; \tag{33}$$

$$\mathbf{I}^{1/2}\mathbf{D}^{1/2} f = f. \tag{34}$$

Relations (33), (34) can be represented symbolically as:

$$\mathbf{D}^{1/2}\mathbf{I}^{1/2}\Big|_{\mathbb{F}} = \mathbb{1}; \tag{35}$$

$$\mathbf{I}^{1/2}\mathbf{D}^{1/2}\Big|_{^C\mathbb{F}} = \mathbb{1}, \tag{36}$$

where $\mathbb{1}$ denotes the identity operator.

Corollary 3.2 shows that Riemann-Liouville integration and differentiation operators define a bijection between linear spaces (20) and (21):

$$\mathbf{D}^{1/2} : {}^C\mathbb{F} \leftrightarrow \mathbb{F} : \mathbf{I}^{1/2}. \tag{37}$$

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathbf{D}^{1/2} : {}^{C}\mathbb{F} \arrow[r, leftrightarrow] \arrow[d, shift right, "{}^{C}\mathbf{I}^{1/2}"'] \arrow[d, shift left, "{}^{C}\mathbf{D}^{1/2}"] & \mathbb{F} : \mathbf{I}^{1/2} \arrow[d, shift right, "\mathbf{I}^{1/2}"'] \arrow[d, shift left, "\mathbf{D}^{1/2}"] \\
\mathbf{D}^{1/2} : {}^{C}\mathbb{F} \arrow[r, leftrightarrow] & \mathbb{F} : \mathbf{I}^{1/2}
\end{tikzcd}$$

Fig. 1. Relationship between Caputo and Riemann-Liouville fractional differentiation operators.

3.5. *Fractional differentiation and integration operators in Caputo linear space*

Fractional differentiation and integration operators ${}^{C}\mathbf{D}^{1/2}, {}^{C}\mathbf{I}^{1/2}$ in Caputo linear space are defined using Riemann-Liouville operators (25), (26) as depicted in Fig. 1.

Note that for any $f \in {}^{C}\mathbb{F}$ the following equality holds true:

$$\mathbf{D}^{1/2}\,{}^{C}\mathbf{D}^{1/2} f = \left(\mathbf{D}^{1/2}\right)^2 f. \tag{38}$$

Corollary 3.2 yields that:

$${}^{C}\mathbf{D}^{1/2} = \mathbf{I}^{1/2}\left(\mathbf{D}^{1/2}\right)^2\Big|_{{}^{C}\mathbb{F}}. \tag{39}$$

Analogously,

$${}^{C}\mathbf{I}^{1/2} = \mathbf{I}^{1/2}\Big|_{{}^{C}\mathbb{F}}. \tag{40}$$

Definition 3.2. Operators ${}^{C}\mathbf{D}^{1/2}, {}^{C}\mathbf{I}^{1/2}$ (defined in (39) and (40)) are called Caputo fractional differentiation and integration operators respectively.

Remark 3.1. Note that Riemann-Liouville and Caputo derivatives coincide for base functions z_j; $j = 2, 3, \ldots$:

$${}^{C}\mathbf{D}^{1/2} z_j = \mathbf{D}^{1/2} z_j = z_{j-1}; \quad j = 2, 3, \ldots \tag{41}$$

However, Riemann-Liouville and Caputo derivatives of $z_1 = 1$ are not equal:

$${}^{C}\mathbf{D}^{1/2} 1 = 0; \qquad \mathbf{D}^{1/2} 1 = \frac{x^{-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)} = z_0. \tag{42}$$

Corollary 3.3. *Let $f \in {}^{C}\mathbb{F}$, $\widehat{f} = \mathbf{D}^{1/2} f$ and $n, m = 0, 1, \ldots$. Then the following relations hold true:*

1. $\mathbf{I}^{1/2}\left(\mathbf{D}^{1/2}\right)^n \widehat{f} = \left({}^{C}\mathbf{D}^{1/2}\right)^n f;$
2. $\mathbf{I}^{1/2}\left(\mathbf{I}^{1/2}\right)^n \widehat{f} = \left({}^{C}\mathbf{I}^{1/2}\right)^n f;$
3. $\mathbf{D}^{1/2}\left({}^{C}\mathbf{D}^{1/2}\right)^n f = \left(\mathbf{D}^{1/2}\right)^n \widehat{f};$
4. $\mathbf{D}^{1/2}\left({}^{C}\mathbf{I}^{1/2}\right)^n f = \left(\mathbf{I}^{1/2}\right)^n \widehat{f};$
5. $\mathbf{I}^{1/2}\left(\mathbf{D}^{1/2}\right)^n\left(\mathbf{I}^{1/2}\right)^m \widehat{f} = \left({}^{C}\mathbf{D}^{1/2}\right)^n\left({}^{C}\mathbf{I}^{1/2}\right)^m f;$
6. $\mathbf{D}^{1/2}\left({}^{C}\mathbf{D}^{1/2}\right)^n\left({}^{C}\mathbf{I}^{1/2}\right)^m f = \left(\mathbf{D}^{1/2}\right)^n\left(\mathbf{I}^{1/2}\right)^m \widehat{f}.$

Proof. *Let $n = 2$, then (35) and (39) yield:*

$$\begin{aligned}
\mathbf{I}^{1/2}\left(\mathbf{D}^{1/2}\right)^2 \widehat{f} \quad &= \quad \mathbf{I}^{1/2}\mathbf{D}^{1/2}\left(\mathbf{D}^{1/2}\mathbf{I}^{1/2}\right)\mathbf{D}^{1/2}\left(\mathbf{D}^{1/2} f\right) \\
&= \quad \left(\mathbf{I}^{1/2}\left(\mathbf{D}^{1/2}\right)^2\right)^2 f = \left({}^{C}\mathbf{D}^{1/2}\right)^2 f.
\end{aligned} \tag{43}$$

The cases $n = 3, 4, \ldots$ and parts (i)-(vi) are proven analogously. □

3.6. *Differentiation and integration operators in Leibniz linear space*

Conventional Leibniz differentiation and integration operators **D, I** read:

$$\mathbf{D}x^n := \begin{cases} 0, & n = 0; \\ n x^{n-1}, & n = 1, 2, \ldots \end{cases} \qquad \mathbf{I}x^n = \frac{x^{n+1}}{n+1}, \quad n = 0, 1, \ldots \tag{44}$$

Leibniz integration and differentiation is defined for base functions $z_1, z_3, \ldots$:

$$\mathbf{D}z_1 = 0, \quad \mathbf{D}z_{2k+1} = z_{2k-1}, \quad k = 1, 2, \ldots \tag{45}$$

$$\mathbf{I}z_{2k+1} = z_{2k+3}, \quad k = 0, 1, \ldots. \tag{46}$$

Furthermore, Leibniz, Riemann-Liouville and Caputo differentiation and integration operators satisfy the following relations:

$$\mathbf{D} = \left(\mathbf{D}^{1/2}\right)^2\Big|_{\mathbb{F}_0} = \left({}^{C}\mathbf{D}^{1/2}\right)^2\Big|_{\mathbb{F}_0}; \tag{47}$$

$$\mathbf{I} = \left(\mathbf{I}^{1/2}\right)^2\Big|_{\mathbb{F}_0} = \left({}^{C}\mathbf{I}^{1/2}\right)^2\Big|_{\mathbb{F}_0}. \tag{48}$$

Remark 3.2. Let $f_0, g_0 \in \mathbb{F}_0$. It follows from (19)–(21) that $f_0 + {}^{C}\mathbf{I}^{1/2} g_0 \in {}^{C}\mathbb{F}$ and $f_0 + \mathbf{D}^{1/2} g_0 \in \mathbb{F}$, thus:

$${}^{C}\mathbb{F} = \mathbb{F}_0 \oplus {}^{C}\mathbf{I}^{1/2}\mathbb{F}_0, \quad \mathbb{F} = \mathbb{F}_0 \oplus \mathbf{D}^{1/2}\mathbb{F}_0. \tag{49}$$

3.7. *Leibniz and Caputo algebras*

3.7.1. *Caputo algebra*

The linear space over $\mathbb{C}$ defined on set (20) can be extended into an algebra over $\mathbb{C}$ by defining the bilinear product between elements of ${}^{C}\mathbb{F}$ in the Cauchy sense. Let $f_1 = \sum_{k=1}^{+\infty} a_k z_k$, $f_2 = \sum_{l=1}^{+\infty} b_l z_l \in {}^{C}\mathbb{F}$. Then the commutative, associative and distributive bilinear product is defined as follows:

$$f_1 f_2 := \sum_{n=2}^{+\infty}\left(\sum_{j=1}^{n-1} a_j b_{n-j}\binom{\frac{n}{2}-1}{\frac{j-1}{2}}\right) z_{n-1}, \tag{50}$$

where

$$z_k z_l = \binom{\frac{k+l}{2}-1}{\frac{k-1}{2}} z_{k+l-1} = \binom{\frac{k+l}{2}-1}{\frac{l-1}{2}} z_{k+l-1}. \tag{51}$$

Note that $\binom{\lambda}{\mu}$ is the generalized binomial coefficient for $\lambda, \mu \in \mathbb{C}$:

$$\binom{\lambda}{\mu} := \frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)\Gamma(\lambda-\mu+1)}. \tag{52}$$

The neutral element with respect to the product is $z_1 = 1$:

$$z_1 f = f z_1 = f, \quad \forall f \in {}^{C}\mathbb{F}. \tag{53}$$

3.7.2. *Leibniz algebra*

For the linear space comprised of Leibniz series (19), the bilinear product is defined analogously as in (50). Let $f_1 = \sum_{k=0}^{+\infty} a_{2k+1} z_{2k+1}$,
$f_2 = \sum_{l=0}^{+\infty} b_{2l+1} z_{2l+1}$, then:

$$f_1 f_2 := \sum_{n=0}^{+\infty}\left(\sum_{j=0}^{n}\binom{n}{j} a_{2j+1} b_{2(n-j)+1}\right) z_{2n+1}. \tag{54}$$

Since product (54) is defined equivalently to (50), the product operation in Leibniz series has the same properties as the product operation for Caputo series.

The Leibniz linear space over $\mathbb{C}$ together with the product (54) forms a sub-algebra of the Caputo algebra defined in the previous subsection.

3.8. Riemann-Liouville algebra

The product between Riemann-Liouville series cannot be defined using (50) – its leading term would have power x^{-1} – which means that the product would not belong to $\mathbb{F}$.

Eq. (37) defines an isomorphism between $\mathbb{F}$ and ${}^C\mathbb{F}$, thus the product operation $*$ between $\widehat{f}_1 = \sum_{k=0}^{+\infty} a_k z_k$ and $\widehat{f}_2 = \sum_{l=0}^{+\infty} b_l z_l \in \mathbb{F}$ can be defined as:

$$\widehat{f}_1 * \widehat{f}_2 := \mathbf{D}^{1/2}\Big(\big(\mathbf{I}^{1/2}\widehat{f}_1\big)\big(\mathbf{I}^{1/2}\widehat{f}_2\big)\Big). \tag{55}$$

Eq. (55) yields the product of the base functions z_k:

$$z_k * z_l = \mathbf{D}^{1/2}(z_{k+1}z_{l+1}) = \binom{\frac{k+l}{2}}{\frac{k}{2}} z_{k+l} = \binom{\frac{k+l}{2}}{\frac{l}{2}} z_{k+l}. \tag{56}$$

Thus (55) can be rewritten as:

$$\widehat{f}_1 * \widehat{f}_2 = \sum_{n=0}^{+\infty}\left(\sum_{j=0}^{n}\binom{\frac{n}{2}}{\frac{j}{2}} a_j b_{n-j}\right) z_n. \tag{57}$$

The neutral element with respect to the $*$ operation is z_0, since (55) and (56) yield that:

$$z_0 * \widehat{f} = \widehat{f} * z_0 = \widehat{f}, \quad \widehat{f} \in \mathbb{F}. \tag{58}$$

Operation (55) shares its properties with operation (50), thus it is associative, commutative and distributive with respect to the sum operation and the product by a scalar operation. Furthermore if $f_k \in {}^C\mathbb{F}, \widehat{f}_k \in \mathbb{F}; k = 1, \dots, n,$ then the following relations hold true:

$$\mathbf{I}^{1/2}\big(\widehat{f}_1 * \widehat{f}_2 * \cdots * \widehat{f}_k\big) = \prod_{k=1}^{n} \mathbf{I}^{1/2}\widehat{f}_k; \tag{59}$$

$$\mathbf{D}^{1/2}\left(\prod_{k=1}^{n} f_k\right) = \big(\mathbf{D}^{1/2}f_1\big) * \big(\mathbf{D}^{1/2}f_2\big) * \cdots * \big(\mathbf{D}^{1/2}f_k\big). \tag{60}$$

Definition of the bilinear product (55) enables the extension of Riemann-Liouville linear space into Riemann-Liouville algebra.

3.9. Agreement of the presented technique with known methods

Note that the solutions obtained for linear fractional differential equations using operators defined in Section 3 are analogous to solutions obtained in literature.

Fractional Malthus equation. Let us consider one of the simplest fractional differential equations – the fractional Malthus exponential growth equation:

$$\begin{aligned} &{}^C\mathbf{D}^{1/2}y = ky, \quad y \in {}^C\mathbb{F}; \quad k \in \mathbb{R}; \\ &y|_{x=0} = \alpha \in \mathbb{R}. \end{aligned} \tag{61}$$

We will demonstrate that the solution to (61) obtained using the presented technique coincides with the solution constructed in [44]:

$$y = \alpha E_{1/2}\big(k\sqrt{x}\big), \tag{62}$$

where $E_\beta(t)$ is the Mittag-Leffler function[45]:

$$E_\beta(t) := \sum_{j=0}^{+\infty} \frac{t^j}{\Gamma(j\beta + 1)}. \tag{63}$$

Let us consider the solution $y \in {}^C\mathbb{F}$ in series form:

$$y = \sum_{j=1}^{+\infty} b_j z_j. \tag{64}$$

The initial condition of (61) is equivalent to:

$$b_1 = \alpha. \tag{65}$$

Inserting (64) into (61) and simplifying yields:

$$\sum_{j=1}^{+\infty} b_{j+1}z_j = k\sum_{j=1}^{+\infty} b_j z_j, \tag{66}$$

which results in the following recursion relation:

$$b_1 = \alpha; \quad b_{j+1} = kb_j; \quad j = 1, 2, \dots. \tag{67}$$

Inserting $b_j = \alpha k^{j-1}, j = 1, 2, \dots$ into (64) yields:

$$y = \alpha \sum_{j=1}^{+\infty} k^{j-1}z_j = \alpha\sum_{s=0}^{+\infty} k^s z_{s+1} = \alpha\sum_{s=0}^{+\infty}\frac{k^s x^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}+1\right)}, \tag{68}$$

which is identical to (62).

Fractional logistic equation. The authors of [44] also consider the fractional logistic equation:

$$\begin{aligned} &{}^C\mathbf{D}^{1/2}y = k(1-y), \quad y \in {}^C\mathbb{F}; \quad k \in \mathbb{R}; \\ &y|_{x=0} = \alpha \in \mathbb{R}. \end{aligned} \tag{69}$$

The solution to (69) given in [44] reads:

$$y = 1 - (1-\alpha)E_{1/2}\big(k\sqrt{x}\big). \tag{70}$$

We will show that the operator scheme can be used to obtain solution (70). Let $y = \sum_{j=1}^{+\infty} b_j z_j \in {}^C\mathbb{F}$. Inserting the series solution into (69) and simplifying yields:

$$\sum_{j=1}^{+\infty} b_{j+1}z_j = k(1-b_1) - \sum_{j=2}^{+\infty} kb_j z_j. \tag{71}$$

Since $b_1 = \alpha$ due to the initial condition, (71) yields the recurrence relations:

$$b_{j+1} = -kb_j, \quad j = 1, 2, \dots, \tag{72}$$

which admit the following solution:

$$b_1 = \alpha; \quad b_{2j} = (1-\alpha)k^{2j-1}; \quad b_{2j+1} = (1-\alpha)k^{2j}; \quad j = 1, 2, \dots. \tag{73}$$

Recurrences (73) yield the solution to (69):

$$\begin{aligned} y &= \sum_{j=1}^{+\infty} b_j z_j = \alpha + (1-\alpha)\sum_{j=1}^{+\infty}\frac{k^{2j-1}x^{\frac{2j-1}{2}}}{\Gamma\left(\frac{2j+1}{2}\right)} - (1-\alpha)\sum_{j=1}^{+\infty}\frac{k^{2j}x^j}{\Gamma(j+1)} \\ &= \alpha + (1-\alpha)\left(\frac{1}{k\sqrt{x}}\sum_{j=1}^{+\infty}\frac{\big(k\sqrt{x}\big)^{2j}}{\Gamma\big(j+\frac{1}{2}\big)} - \sum_{j=1}^{+\infty}\frac{\big(k^2x\big)^j}{\Gamma(j+1)}\right) \\ &= \alpha + (1-\alpha)\Big(\exp\big(k^2x\big)\operatorname{erf}\big(k\sqrt{x}\big) + \exp\big(k^2x\big) - 1\Big) \\ &= 1 - (1-\alpha)\exp\big(k^2x\big)\Big(1 + \operatorname{erf}\big(k\sqrt{x}\big)\Big), \end{aligned} \tag{74}$$

where $\operatorname{erf}(t)$ denotes the error function [46]:

$$\operatorname{erf}(t) := \frac{2}{\sqrt{\pi}}\int_0^t \exp\big(-\tau^2\big)\,\mathrm{d}\tau; \quad t > 0. \tag{75}$$

In derivation (74), the following identity was used [46]:

$$\sum_{j=1}^{+\infty}\frac{t^{2j}}{\Gamma\big(j+\frac{1}{2}\big)} = t\exp\big(t^2\big)\operatorname{erf}(t); \quad t > 0. \tag{76}$$

Note that the error function and the Mittag-Leffler function are related by the following equation [47]:

$$E_{1/2}(t) = \exp\big(t^2\big)(1 + \operatorname{erf}(t)), \tag{77}$$

thus (74) can be written in a form identical to (70).

4. Construction of analytical solutions to the fractional Riccati equation (1)

Let $y = \sum_{j=1}^{+\infty} b_j z_j \in {}^C\mathbb{F}; \quad b_j \in \mathbb{R}$. Eq. (1) can be rewritten using base functions (8):

$$z_{k+1}\left({}^C\mathbf{D}^{1/2}\right)^k y = A_0 + A_1 y + A_2 y^2, \tag{78}$$

where $A_j = \frac{B_j}{\Gamma\left(\frac{k}{2}+1\right)}$, $j = 0, 1, 2$. The initial condition on (78) reads:

$$^C\mathbf{D}^{1/2} y\Big|_{x=0} = \alpha \in \mathbb{R}, \tag{79}$$

which is equivalent to:

$$b_2 = \alpha. \tag{80}$$

Note that it is not possible to construct the solution of (78) in Caputo algebra using standard operator techniques due to the fact that the additivity of base function powers under multiplication is not preserved. However, such non-symmetrical behavior is not present in Riemann-Liouville algebra, because $z_k * z_l = \binom{\frac{k+l}{2}}{\frac{k}{2}} z_{k+l}$. Thus instead of (78), the equivalent differential equation can be considered in Riemann-Liouville algebra.

Denoting

$$\widehat{y} = \mathbf{D}^{1/2} y = \sum_{j=0}^{+\infty} c_j z_j, \quad c_j := b_{j+1}, \ j = 0, 1, \ldots; \tag{81}$$

and applying the operator $\mathbf{D}^{1/2}$ to both sides of (78) results in:

$$z_k * \left(\mathbf{D}^{1/2}\right)^k \widehat{y} = A_0 z_0 + A_1 \widehat{y} + A_2 \widehat{y} * \widehat{y}. \tag{82}$$

The initial condition (79) is transformed into:

$$\mathbf{I}^{1/2}\mathbf{D}^{1/2}\widehat{y}\Big|_{x=0} = c_1 = \alpha. \tag{83}$$

Note that (83) is equivalent to (80).

Relation between solutions to (78) and (82) reads:

$$y = \mathbf{I}^{1/2}\widehat{y}. \tag{84}$$

4.1. Case $k = 1$

The fractional Riccati equation for $k = 1$ reads:

$$\frac{\sqrt{x}}{\Gamma\left(\frac{3}{2}\right)} {}^C\mathbf{D}^{1/2} y = A_0 + A_1 y + A_2 y^2; \tag{85}$$

and

$$1 * \mathbf{D}^{1/2}\widehat{y} = A_0 z_0 + A_1 \widehat{y} + A_2 \widehat{y} * \widehat{y}, \tag{86}$$

in Caputo and Riemann-Liouville algebras respectively. Initial conditions on (85), (86) are (79), (83) respectively.

4.1.1. Construction of recurrence relations for solution parameters

Inserting the expression $\widehat{y} = \sum_{j=0}^{+\infty} c_j z_j$ into (86) yields:

$$1 * \mathbf{D}^{1/2}\left(\sum_{j=0}^{+\infty} c_j z_j\right) = A_2 \left(\sum_{r=0}^{+\infty} c_r z_r\right) * \left(\sum_{l=0}^{+\infty} c_l z_l\right) + A_1 \sum_{k=0}^{+\infty} c_k z_k + A_0 z_0. \tag{87}$$

The relation $z_1 = 1$ and (56), (57) results in:

$$\sum_{j=1}^{+\infty} \binom{j/2}{1/2} c_j z_j = A_2 \sum_{j=0}^{+\infty}\sum_{k=0}^{j} \binom{j/2}{k/2} c_k c_{j-k} z_j + A_1 \sum_{k=0}^{+\infty} c_k z_k + A_0 z_0. \tag{88}$$

Gathering like terms on both sides of (88) results in the following system of nonlinear recurrence relations:

$$A_2 c_0^2 + A_1 c_0 + A_0 = 0; \tag{89}$$

$$c_1 = c_1(2A_2 c_0 + A_1); \tag{90}$$

$$\binom{m/2}{1/2} c_m = A_2 \sum_{k=1}^{m-1} \binom{m/2}{k/2} c_k c_{m-k} + A_1 c_m + 2A_2 c_m c_0, \quad m = 2, 3, \ldots. \tag{91}$$

Eqs. (89), (90) and (91) correspond to the cases $j = 0$, $j = 1$ and $j = 2, 3, \ldots$ respectively.

Note that choosing $c_1 = 0$ satisfies (90) and results in the following trivial solution:

$$\widehat{y} = c_0 z_0 = \frac{c_0}{\sqrt{\pi x}}, \tag{92}$$

where c_0 satisfies (89). In subsequent derivations it is assumed that $c_1 \neq 0$.

4.1.2. Existence condition for $\widehat{y}$ in (86)

The parameter c_0 must satisfy (89) if $\widehat{y}$ satisfies (87). Because $c_1 \neq 0$, the solution to (90) with respect to c_0 reads:

$$c_0 = \frac{1 - A_1}{2A_2}. \tag{93}$$

The parameter c_0 must satisfy (89), thus (93) and (89) yield the existence condition for solution $\widehat{y}$ in (86):

$$A_1^2 - 4A_0 A_2 = 1. \tag{94}$$

4.1.3. Solution of (91)

Note $A_1 + 2A_2 c_0 = 1$ from (90). Solving (91) with respect to c_m yields:

$$c_m = \frac{A_2}{\binom{m/2}{1/2} - 1} \sum_{k=1}^{m-1} \binom{m/2}{k/2} c_k c_{m-k} \tag{95}$$

where $m = 2, 3, \ldots$.

Multiplying (95) by $\frac{A_2}{\Gamma\left(\frac{m}{2}+1\right)}$ yields:

$$\frac{A_2 c_m}{\Gamma\left(\frac{m}{2}+1\right)} = \frac{1}{\binom{m/2}{1/2} - 1} \sum_{k=1}^{m-1} \frac{A_2 c_k}{\Gamma\left(\frac{k}{2}+1\right)} \frac{A_2 c_{m-k}}{\Gamma\left(\frac{m-k}{2}+1\right)}, \quad m = 2, 3, \ldots \tag{96}$$

Denoting $\gamma_m := \frac{A_2 c_m}{\Gamma\left(\frac{m}{2}+1\right)}$, $m = 1, 2, \ldots$ results in the sequence:

$$\gamma_m = \frac{1}{\binom{m/2}{1/2} - 1} \sum_{k=1}^{m-1} \gamma_k \gamma_{m-k}, \quad m = 2, 3, \ldots. \tag{97}$$

Let us consider the sequence $\nu_k^{(1)}$, $k = 1, 2, \ldots$ with initial condition $\nu_1^{(1)} = 1$ that has the following structure:

$$\nu_m^{(1)} = \frac{1}{\binom{m/2}{1/2} - 1} \sum_{k=1}^{m-1} \nu_k^{(1)} \nu_{m-k}^{(1)}, \quad m = 2, 3, \ldots. \tag{98}$$

Note that (98) is satisfied by the sequence $\nu_{m+1}^{(1)}\rho^{m+1}$, $m = 0, 1, \ldots; \rho \in \mathbb{R}$:

$$\nu_m^{(1)}\rho^m = \frac{1}{\binom{m/2}{1/2} - 1} \sum_{k=1}^{m-1} \left(\nu_k^{(1)}\rho^k\right)\left(\nu_{m-k}^{(1)}\rho^{m-k}\right), \quad m = 2, 3, \ldots. \tag{99}$$

Thus the solution to recurrence (97) is $\gamma_m = \nu_m^{(1)}\rho^m$, $m = 1, 2, \ldots$, where $\nu_m^{(1)}$ satisfies (98), $\nu_1^{(1)} = 1$ and $\rho \in \mathbb{R}$.

4.1.4. Construction of analytical solutions to the fractional Riccati equation

The results of Section 4.1.2–4.1.3 yield the coefficients c_j:

$$c_0 = \frac{1 - A_1}{2A_2}, \quad c_j = \frac{\Gamma\left(\frac{j}{2} + 1\right)}{A_2} \nu_j^{(1)} \rho^j, \ j = 1, 2, \ldots, \tag{100}$$

where $\rho \in \mathbb{R}$ and $\nu_j^{(1)}$ satisfies (98).

Relation (100) with $j = 1$ yields that the initial condition (83) is satisfied if:

$$\frac{\rho\Gamma\left(\frac{3}{2}\right)}{A_2} = \alpha; \quad \alpha \in \mathbb{R}, \tag{101}$$

thus

$$\rho = \frac{2A_2\alpha}{\sqrt{\pi}}, \quad \alpha \in \mathbb{R}. \tag{102}$$

Eqs. (100) and (102) yield the analytical solution to (86):

$$\widehat{y} = \frac{1 - A_1}{2A_2\sqrt{\pi x}} + \frac{2}{A_2\sqrt{\pi}} \sum_{j=1}^{+\infty} \binom{j/2}{1/2} \nu_j^{(1)} \left(\frac{2A_2\alpha}{\sqrt{\pi}}\right)^j x^{\frac{j-1}{2}}. \tag{103}$$

Theorem 4.1. *Analytical solution (103) satisfies the Cauchy problem (86), (83) if and only if $A_2 \neq 0$, $\alpha \neq 0$ and (94) hold true.*

Proof. *Derivations given in Section 4.1.1–4.1.3 together with (101), (102) ensure necessity and sufficiency.* □

Corollary 4.1. *Analytical solution*

$$y = \frac{1 - A_1}{2A_2} + \frac{2}{A_2\pi} \sum_{j=1}^{+\infty} j\nu_j^{(1)} \left(\frac{2A_2\alpha}{\sqrt{\pi}}\right)^j x^{\frac{j}{2}}; \tag{104}$$

satisfies the Cauchy problem (85), (79) if and only if $A_2 \neq 0$, $\alpha \neq 0$ and (94) hold true.

Proof. *The Cauchy problem (86), (83) is obtained by setting $\widehat{y} = \mathbf{D}^{1/2}y$ in (85), (79). Relation (36) results in $y = \mathbf{I}^{1/2}\widehat{y}$, thus*

$$\begin{aligned}
\binom{j/2}{1/2}\mathbf{I}^{1/2}x^{\frac{j-1}{2}} &= \binom{j/2}{1/2}\frac{\Gamma\left(\frac{j+1}{2}\right)}{\Gamma\left(\frac{j}{2}\right)}x^{\frac{j}{2}} = \frac{\Gamma\left(\frac{j}{2}+1\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{j+1}{2}\right)}\frac{\Gamma\left(\frac{j+1}{2}\right)}{\Gamma\left(\frac{j}{2}\right)}x^{\frac{j}{2}} \\
&= \frac{2}{\sqrt{\pi}}\frac{\Gamma\left(\frac{j}{2}+1\right)}{\Gamma\left(\frac{j}{2}\right)}x^{\frac{j}{2}} = \frac{j}{\sqrt{\pi}}x^{\frac{j}{2}}.
\end{aligned} \tag{105}$$

Furthermore, integrating both sides of (86) yields (85). It has already been shown in Section 4 that initial conditions (79) and (83) are equivalent.

Thus (104) satisfies (85), (79) if and only if the conditions of Theorem 4.1 hold true. □

4.2. Case $k = 2$

Let us consider (78) in the case $k = 2$:

$$x\left({}^C\mathbf{D}^{1/2}\right)^2 y = A_0 + A_1 y + A_2 y^2. \tag{106}$$

It follows from (82) that in Riemann-Liouville algebra (106) reads:

$$z_2 * \left(\mathbf{D}^{1/2}\right)^2 \widehat{y} = A_0 z_0 + A_1 \widehat{y} + A_2 \widehat{y} * \widehat{y}. \tag{107}$$

Eqs. (106) and (107) are subject to initial condition (80).

4.2.1. Existence conditions for $\widehat{y}$ in (106)

Inserting the solution $\widehat{y} = \sum_{j=0}^{+\infty} c_j z_j$ into (107) and gathering like terms results in the following relations:

$$A_2 c_0^2 + A_1 c_0 + A_0 = 0; \tag{108}$$

$$c_1(2A_2c_0 + A_1) = 0; \tag{109}$$

$$\frac{m}{2}c_m = A_2 \sum_{k=0}^{m} \binom{m/2}{k/2} c_k c_{m-k} + A_1 c_m, \quad m = 2, 3, \ldots. \tag{110}$$

The case $c_1 = 0$ results in $c_m = 0, m = 2, 3, \ldots$ and yields the trivial solution

$$\widehat{y} = \frac{c_0}{\sqrt{\pi x}}, \tag{111}$$

where c_0 satisfies (108).

Let $c_1 \neq 0$. Expressions (108) and (109) yield the existence conditions for $\widehat{y}$ in (106):

$$c_0 = -\frac{A_1}{2A_2}, \quad A_2 \neq 0; \tag{112}$$

$$A_1^2 - 4A_2A_0 = 0. \tag{113}$$

4.2.2. Solution of (110)

The algorithm described in Sections 4.1 and 4.1.3 yields that the solution to recurrence relation (110) reads:

$$c_m = \frac{\Gamma\left(\frac{m}{2} + 1\right)}{2A_2} \nu_m^{(2)} \rho^m, \quad m = 1, 2, 3, \ldots, \tag{114}$$

where $\rho \in \mathbb{R}$ and $\nu_m^{(2)}$ satisfies:

$$\nu_1^{(2)} = 1, \quad \nu_m^{(2)} = \frac{1}{m}\sum_{k=1}^{m-1} \nu_k^{(2)}\nu_{m-k}^{(2)}, \quad m = 2, 3, \ldots. \tag{115}$$

4.2.3. Construction of solution to (106)

Initial condition (80) yields:

$$\rho = \frac{4A_2\alpha}{\sqrt{\pi}}, \tag{116}$$

thus (112) and (114) yield the analytical solution to (107):

$$\widehat{y} = -\frac{A_1}{2A_2\sqrt{\pi x}} + \frac{1}{A_2\sqrt{\pi}} \sum_{j=1}^{+\infty} \binom{j/2}{1/2} \nu_j^{(2)} \left(\frac{4A_2\alpha}{\sqrt{\pi}}\right)^j x^{\frac{j-1}{2}}. \tag{117}$$

Theorem 4.2. *Analytical solution (117) satisfies (107) with initial condition (83) if and only if $A_2 \neq 0$, $\alpha \neq 0$ and (113) holds true.*

Proof. *Proof is given by the preceding derivations.* □

Corollary 4.2. *Analytical solution*

$$y = -\frac{A_1}{2A_2} + \frac{1}{A_2\pi} \sum_{j=1}^{+\infty} j\nu_j^{(2)} \left(\frac{4A_2\alpha}{\sqrt{\pi}}\right)^j x^{\frac{j}{2}}. \tag{118}$$

satisfies (106) with initial condition (79) if and only if $A_2 \neq 0$, $\alpha \neq 0$ and (113) holds true.

Proof. *Proof is analogous to Corollary (4.1).* □

4.2.4. Construction of closed-form solution to (78)

Sequences and generating functions. Let us consider the real-valued sequences $P = (p_0, p_1, \ldots)$ and $Q = (q_0, q_1, \ldots)$. Standard sum, multiplication by scalar and convolution operations are defined for P, Q [48]:

$$P + Q = (p_0 + q_0, p_1 + q_1, \ldots); \tag{119}$$

$$\beta P = (\beta p_0, \beta p_1, \ldots), \quad \beta \in \mathbb{R}; \tag{120}$$

$$P * Q = (p_0q_0, p_1q_0 + p_0q_1, p_2q_0 + p_1q_1 + p_0q_2, \ldots). \tag{121}$$

The ordinary generating function for sequence P is defined as [48]:

$$w_P(t) := \sum_{j=0}^{+\infty} p_j t^j. \tag{122}$$

In subsequent derivations the following properties of generating functions will be used:

$$w_{\beta P+\delta Q} = \beta w_P + \delta w_Q, \quad \beta, \delta \in \mathbb{R}; \tag{123}$$

$$w_{P*Q} = w_P w_Q. \tag{124}$$

Generating functions of sequence (115). Let us denote $\nu_0^{(2)} = 0$ and $S_0 := \left(\nu_0^{(2)}, \nu_1^{(2)}, \dots\right)$. The generating function of sequence S_0 reads:

$$w_{S_0}(t) = \sum_{j=0}^{+\infty} \nu_j^{(2)} t^j. \tag{125}$$

Note that since $\nu_0^{(2)} = 0$, recurrence relation (115) can be rewritten as:

$$m\nu_m^{(2)} = \sum_{k=0}^{m} \nu_k^{(2)} \nu_{m-k}^{(2)}, \quad m = 2, 3, \dots. \tag{126}$$

It follows from (126) that sequence $S_1 := \left(j\nu_j^{(2)};\ j = 0, 1, \dots\right)$ has the following form:

$$\left(j\nu_j^{(2)};\ j = 0, 1, \dots\right) = \left(\nu_0^{(2)}, \nu_1^{(2)}, \nu_2^{(2)}, \dots\right) * \left(\nu_0^{(2)}, \nu_1^{(2)}, \nu_2^{(2)}, \dots\right) + (0, 1, 0, \dots). \tag{127}$$

Note that generating function of $\left(j\nu_j^{(2)};\ j = 0, 1, \dots\right)$ is the derivative of (125):

$$w_{S_1}(t) = \mathbf{D}_t w_{S_0}(t) = \sum_{j=1}^{+\infty} j\nu_j^{(2)} t^{j-1}, \tag{128}$$

thus (127) yields the following ordinary differential equation with respect to $w_{S_0}(t)$:

$$t\mathbf{D}_t w_{S_0} = w_{S_0}^2 + t. \tag{129}$$

Eq. (129) is subject to initial condition $\mathbf{D}_t w_{S_0}\big|_{t=0} = \nu_1^{(2)} = 1$. However, since $\mathbf{D}_t w_{S_0} = \frac{w_{S_0}^2}{t} + 1$, the initial condition can be rewritten as:

$$\left.\frac{w_{S_0}^2}{t}\right|_{t=0} = 0. \tag{130}$$

The general solution to (129) reads:

$$w_{S_0}(t) = \frac{\sqrt{t}\left(CY_1\left(2\sqrt{t}\right) + J_1\left(2\sqrt{t}\right)\right)}{CY_0\left(2\sqrt{t}\right) + J_0\left(2\sqrt{t}\right)}, \quad C \in \mathbb{R}, \tag{131}$$

where $J_\beta(t)$, $Y_\beta(t)$ are Bessel functions of the first and second kind respectively [42]:

$$\begin{aligned} J_\beta(t) &= \sum_{m=0}^{+\infty} \frac{(-1)^m}{m!\,\Gamma(m+\beta+1)} \left(\frac{t}{2}\right)^{2m+\beta}; \\ Y_\beta(t) &= \lim_{\gamma\to\beta} \frac{J_\gamma(t)\cos(\gamma\pi) - J_{-\gamma}(t)}{\sin(\gamma\pi)}. \end{aligned} \tag{132}$$

Solution (131) satisfies initial condition (130) if C satisfies the following relation:

$$CY_1\left(2\sqrt{t}\right) + J_1\left(2\sqrt{t}\right)\Big|_{t=0} = 0. \tag{133}$$

Because the limit of $Y_1(t)$ as t tends to zero is $\pm\infty$, (133) yields that $C = 0$, thus:

$$w_{S_0}(t) = \frac{\sqrt{t}J_1\left(2\sqrt{t}\right)}{J_0\left(2\sqrt{t}\right)}. \tag{134}$$

Closed-form solution to (78). Let us denote $t := \frac{4A_2\alpha}{\sqrt{\pi}}\sqrt{x}$; $a_0 := -\frac{A_1}{2A_2}$; $a_1 := \frac{1}{A_2\pi}$. Then by (118), the solution to (78) reads:

$$y(x) = w_1(t) = a_0 + a_1 \sum_{j=1}^{+\infty} j\nu_j^{(2)} t^j. \tag{135}$$

Because the right side of (135) contains the derivative of the generating function for sequence $\left(\nu_0^{(2)}, \nu_1^{(2)}, \dots\right)$, equations (128), (129) and (134) yield the closed-form solution to (78):

$$\begin{aligned} \tilde{y}(t) &= a_0 + a_1 t\mathbf{D}_t w_{S_0}(t) = a_0 + a_1\left(w_{S_0}^2 + t\right) \\ &= a_0 + a_1 \frac{t\left(J_0^2\left(2\sqrt{t}\right) + J_1^2\left(2\sqrt{t}\right)\right)}{J_0^2\left(2\sqrt{t}\right)}, \end{aligned} \tag{136}$$

where $t = \frac{4A_2\alpha}{\sqrt{\pi}}\sqrt{x}$ and

$$y(x) = \tilde{y}\left(\frac{4A_2\alpha}{\sqrt{\pi}}\sqrt{x}\right). \tag{137}$$

The closed-form solutions to (78) are depicted in Fig. 2.

4.3. Cases $k = 3, 4, \dots$

Eq. (78) for $k = 3, 4, \dots$ only has trivial solutions. For example, for $k = 3$ Eq. (78) reads:

$$z_3 * \left(\mathbf{D}^{1/2}\right)^3 \hat{y} = A_0 z_0 + A_1\hat{y} + A_2\hat{y} * \hat{y}. \tag{138}$$

Inserting the solution $\hat{y} = \sum_{j=0}^{+\infty} c_j z_j$ into (138) yields the following relations:

$$A_2c_0^2 + A_1c_0 + A_0 = 0; \tag{139}$$

$$2A_2c_0 + A_1 = 0; \tag{140}$$

$$2A_2c_0c_2 + A_2c_1^2 + A_1c_2 = 0; \tag{141}$$

$$\binom{m/2}{3/2} c_m = A_2 \sum_{k=1}^{m-1} \binom{m/2}{k/2} c_k c_{m-k} + 2A_2c_0c_m + A_1c_m, \quad m = 3, 4, \dots. \tag{142}$$

Eqs. (139)–(141) can only hold true if $c_0 = -\frac{A_1}{2A_2}$ and $c_m = 0, m = 1, 2, \dots$, thus (138) only has the trivial solution:

$$\hat{y} = -\frac{A_1}{2A_2\sqrt{\pi x}}. \tag{143}$$

The same argument holds true in cases $k = 4, 5, \dots$. Thus non-trivial analytical solutions to (1) are only obtained in the cases $k = 1, 2$.

4.4. Comparison to other techniques

A number of approaches to solving fractional Riccati equation (1) have been described in literature. The Adomian decomposition method is used to obtain approximate solutions to the fractional Riccati equation with Caputo derivative in [49]. Power series obtained by the fractional variational iteration method are used to obtain approximate solutions to fractional Riccati equations with modified Riemann-Liouville fractional derivative in [50]. A generalization of the Adomian decomposition method yields series solutions to nonlinear fractional Riccati equations with Caputo derivative in [51]. The collocation method is used to generate approximate solutions to fractional Riccati-type differential equations in [52].

Techniques presented in this paper are used to obtain analytical closed-form solutions to (1). It is clear that such solution are preferable over approximations and provide deeper insight into fractional Riccati fractional equations.

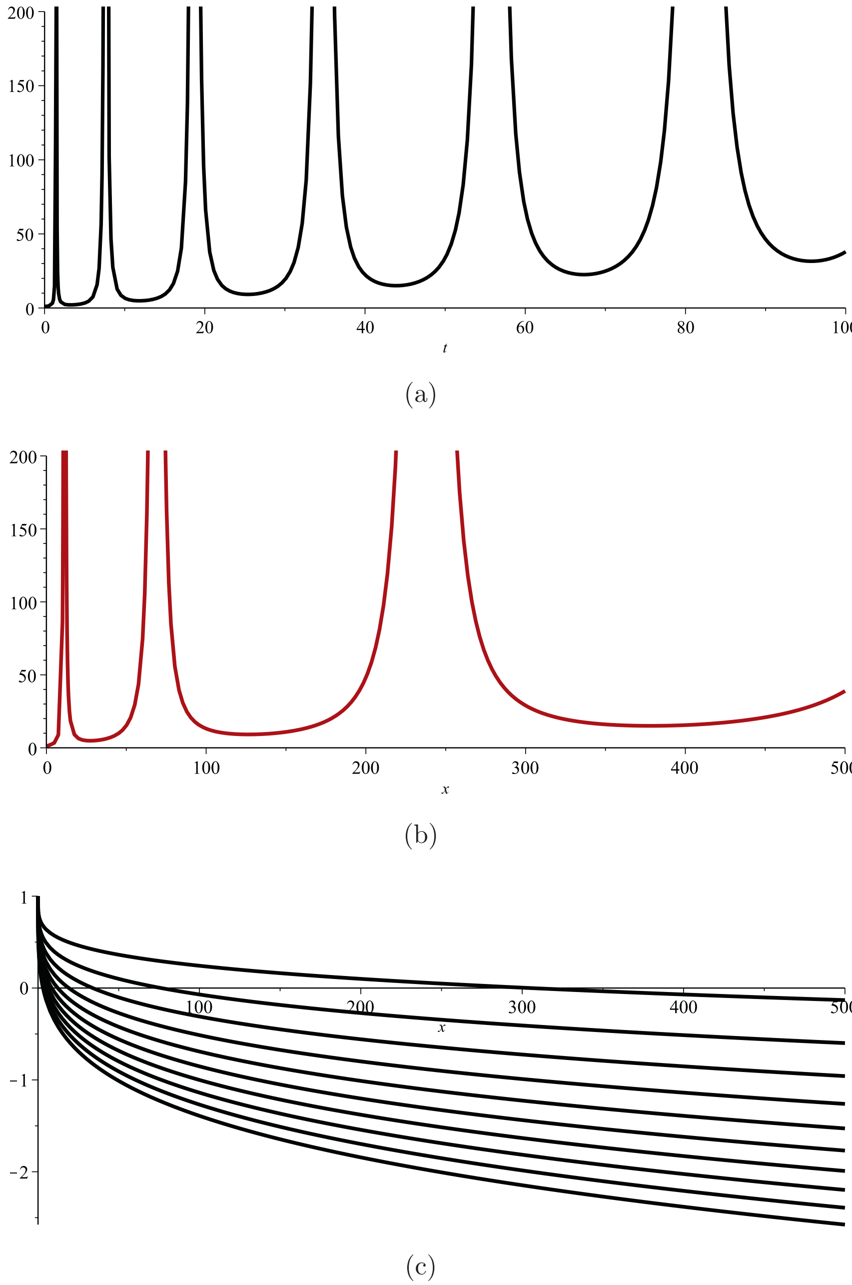

Fig. 2. Closed-form solutions to (78). Part (a) shows the solution (136), which is independent of initial condition α. Parts (b) and (c) depict the explicit closed-form solution (137). Initial condition α is equal to 1 in part (b). Part (c) depict the family of solutions (137) for $\alpha = -10, -9, \ldots, -1$, where the values -10 and -1 correspond to the lowest and highest curves respectively. Note that the singularity points in (a) and (b) correspond to the zeros of the Bessel function in the denominator of (136). For $\alpha < 0$, the solution (137) in nonsingular.

5. Conclusion

An analytical framework for the construction of analytical solutions to Caputo and Riemann-Liouville fractional differential equations is presented in this paper. Caputo fractional differential equations are mapped to Riemann-Liouville equations in order to preserve the additivity of base function powers under multiplication. The proposed scheme is successfully applied to obtain closed-form and power series solutions to a class of fractional Riccati equations.

The aim of this paper is to present a novel technique for the construction of closed-form solutions to fractional differential equations. Because the fractional Riccati equation is one of the simplest nonlinear fractional models it provides a clear illustration of the considered technique. The ability to construct closed-form solutions to fractional differential equations reveals new possibilities in analytical investigation of fractional models in nonlinear science.

The variational approach for the derivation of fractional differential equations is an important element for the physical analysis of the considered model. Extensions of traditional calculus of variations for systems containing fractional derivatives have been presented in [53] and [54]. Euler-Lagrange type equations which depend on the Riemann-Liouville fractional derivative are derived using calculus of variations in [55].

Even though fractional Riccati equation (1) is not introduced using a variational approach, its value stems from the theory of nonlinear evolutions (with integer derivatives) which do not require calculus of variations (for example the Korteweg-de-Vries equation

[56]). To construct solutions to such equations, a coordinate transformation is used to transform nonlinear partial differential equations to ordinary differential equations. It is demonstrated that direct solution construction methods do not yield closed-form solutions, because the sequence of power series coefficients do not form a linear recurrence. A nonlinear transformation is then used to construct an image equation, to which various solitary solutions can be constructed using operator techniques. Finally, reverse transformation yields closed-form solitary solutions to the original equation [57]. This paper is a first attempt to apply the described operator approach to fractional differential equations for the construction of solitary solutions, which do have physical significance to nonlinear evolution problems.

The presented operator technique is a solid foundation for the analysis of series and closed-form solutions to fractional-order differential equations. Because the main objective of this paper is to present a new technique for the construction of closed-form solutions, only equations with derivatives of order $\frac{1}{2}$ are considered. However, it is possible to extend the scheme to enable the consideration of any rational derivative order $\frac{m}{n}$; $m, n \in \mathbb{N}$. Such extensions are definite objectives of future research.

Acknowledgment

This research was funded by a grant (No. MIP078/2015) from the Research Council of Lithuania.

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