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How far one can go with the Exp-function method?

Zenonas Navickas [a], Minvydas Ragulskis [b,*]

[a] Department of Applied Mathematics, Kaunas University of Technology, Studentu 50-325, Kaunas LT-51368, Lithuania
[b] Research Group for Mathematical and Numerical Analysis of Dynamical Systems, Kaunas University of Technology, Studentu 50-222, Kaunas LT-51368, Lithuania

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ABSTRACT

A criterion determining if an exact solution of a differential equation can be expressed in a form comprising a finite number of exponential functions is constructed in this paper. This criterion is based on the concept of ranks of Hankel matrixes constructed from sequences of coefficients produced by symbolic multiplicative operator techniques. The employment of this criterion also gives an answer on the structure of the solution. Several examples are used to illustrate this concept.

1. Introduction

The Exp-function method was proposed in 2006 by He and Wu [1] to seek solitary solutions, periodic solutions and compacton-like solutions of nonlinear differential equations. This seminal paper has initiated an explosive reaction in the scientific community. The Exp-function method helped to find exact solutions of many differential equations, which solutions previously were considered not to be expressible in an exact form comprising a finite number of ordinary functions. It has been demonstrated that the Exp-function method, with the help of symbolic computation, provides a powerful mathematical tool for solving high-dimensional nonlinear evolutions in mathematical physics. Few recent references can illustrate that dynamism.

New solitonary solutions are derived for the modified forms of Degasperis–Procesi and Camassa–Holm equations [2]; the effectiveness of the Exp-function method is demonstrated on the KdV equation, Burgers' equation and the combined Korteweg–de Vries KdV–mKdV equation [3]. The Exp-function method is used to obtain exact generalized solitary solutions and periodic solutions for nonlinear improved Boussinesq equation [4]; the Liouville equation [5] and Maccari's system [6]. The Exp-function method with the aid of Maple is used to obtain generalized soliton solution for the symmetric regularized long wave (SRLW) equation [7] and the modified Benjamin–Bona–Mahony equations [8]. The Exp-function method is exploited to construct new generalized soliton solutions for the discrete (2 + 1)-dimensional Toda lattice equation [9], the Jaulent–Miodek equations [10], the Kawahara and modified Kawahara equations [11], the Klein–Gordon, Burger–Fisher and Sharma–Tasso–Olver equations [12], Broer–Kaup–Kupershmidt equations [13], the Korteweg–de Vries–Burgers (KdVB) equation [14], a KdV equation with a forcing term [15], the (3 + 1)-dimensional Kadomtsev–Petviashvili equation [16], variant Boussinesq equations [17].

It is very likely that the flow of papers describing new exact solutions of various nonlinear differential equations will continue in the future. Natural is the question if there does exist an analytical criterion determining if an exact solution of a differential equation (ordinary or partial) can be found by the Exp-function method. The object of this paper is to find and construct such a criterion.

* Corresponding author.
E-mail addresses: zenonas.navickas@ktu.lt (Z. Navickas), minvydas.ragulskis@ktu.lt (M. Ragulskis).
URL: http://www.personalas.ktu.lt/~mragul (M. Ragulskis).

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2. Preliminary considerations

First of all we will give a short overview of the multiplicative operator method for the solution of differential equations in this section. Detailed proofs of the results presented in this section (also some proofs in the third section) are omitted because they are available at provided references. Nevertheless, we mention the very basic facts associated with this method and illustrate it by giving few examples. We feel that this information is necessary for definitions and operations presented in next sections.

2.1. Functions and their expansions

We denote F as a set of real or complex functions $f(x)$. It is assumed that every function $f(x) \in F$ can be expanded in a series of non-negative powers

$$f(x) = \sum_{j=0}^{+\infty} p_j \frac{(x-a)^j}{j!}; \quad x \in D_f, \tag{1}$$

where D_f is an individual region of convergence for every function $f(x)$; coefficients p_j can depend on one or several parameters; $a \in R$. We make a proposition that every function $f(x) \in F$ can be extended from its region of convergence D_f into the whole set of real numbers R or even the complex plane C. This extension can be performed in the one and only way with the exception of a limited number of special points $x_1, \ldots, x_n$ where the following limits hold:

$$\lim_{x \to x_i} |f(x)| = +\infty; \quad i = 1, 2, \ldots, n. \tag{2}$$

For example a function $f(x) = s + \frac{tx}{1!} - \frac{sx^2}{2!} - \frac{tx^3}{3!} + \cdots$ converges at all $x \in C$:$f(x) = s\cos x + t\sin x$ for all $s,t \in R$.

But a series $f(x) = s + s^2x + s^3x^2 + \cdots$ converges only at $|x| < \frac{1}{|s|}$; $s \in R\backslash\{0\}$. Nevertheless, this function can be extended into the whole complex plane C with the exception of a special point $x = \frac{1}{s}$:

$$f(x) = \frac{s}{1-sx}; \quad \lim_{x \to \frac{1}{s}} |f(x)| = +\infty.$$

As mentioned earlier, every function $f(x) \in F$ can be expanded in a power series in the surroundings of a point x if only x is not a special point. The area of convergence is a positive radius sphere around the point x. But the radius of convergence is not important for the further analysis; thus we will not discuss convergence in the next sections.

It can be noted that functions $f(x) \in F$ can be differentiated any number of times (except at their special points); relevant series can be differentiated term after term. For example:

$$\frac{d\left(\sum_{j=0}^{+\infty}(sx)^j\right)}{dx} = \sum_{j=1}^{+\infty} js^jx^{j-1} = s\sum_{j=0}^{+\infty}(j+1)(sx)^j = \frac{s}{(1-sx)^2}. \tag{3}$$

Moreover, other operations (integration, multiplication by another function or a parameter) with functions $f(x) \in F$ and their respective series are correct and will not lead to wrong results [18].

2.2. The generalized operator of differentiation

We will denote linear operations of integration and differentiation by symbols L and D. We will use appropriate indexes to indicate the variable of the operation. For example,

$$L_xD_sx^ms^n = (L_xx^m)(D_ss^n) = \frac{n}{m+1}x^{m+1}s^{n-1}; \quad m,n \in Z_0\left(Z_0 := 0,1,2,\ldots;\ L_xx^m := \int_0^x x^m\,dx = \frac{1}{m+1}x^{m+1}\right).$$

Linear operators L_x and D_s are commutative, but $L_xD_x \neq D_xL_x$. It can be noted that $D_xL_x = 1$. Further on the symbol **1** will stand for the unitary operator and **O** – for the zero operator:

$$1x^n := x^n; \quad \mathbf{O}x^n := 0, \quad n \in Z_0. \tag{4}$$

We will exploit all standard properties of linear operators [18] in further analysis.

Let $P = P(x,s,t)$ and $Q = Q(x,s,t)$ be polynomials or ratios of polynomials of variables x, s and t. Let a function $f_k \in F$ depend on parameters s and t: $f_k = f_k(x;s,t)$, $k = 1,2\ldots,n$. Then a generalized operator of differentiation can be introduced:

$$D_{st} := PD_s + QD_t. \tag{5}$$

The linear operator D_{st} has a number of special properties [19]:

$$D_{st}\sum_{k=1}^{n}\alpha_kf_k = \sum_{k=1}^{n}\alpha_kD_{st}f_k, \quad \alpha_k \in C, \tag{6}$$

$$D_{st}^m(f_k \cdot f_l) = \sum_{j=0}^{m}\binom{m}{k}(D_{st}^jf_k)(D_{st}^{m-j}f_l), \quad m = 0,1,\ldots, \tag{7}$$

where $D_{st}^0 := \mathbf{1}$; $\mathbf{O}^0 := \mathbf{1}$; $\binom{m}{j} := \frac{m!}{(m-j)!j!}$ for $m \geqslant j \geqslant 0$.

$$D_{st} f_k^m = m f_k^{-1}(D_{st} f_k), \tag{8}$$

$$D_{st}\frac{f_k}{f_l} = \frac{(D_{st} f_k) f_l - f_k (D_{st} f_l)}{f_l^2}. \tag{9}$$

2.3. The multiplicative operator

We define the multiplicative operator:

$$G = G(D_{st}) := \sum_{k=0}^{+\infty} (L_x D_{st})^k; \quad ((L_x D_{st})^0 = 1). \tag{10}$$

For example, when $D_{st} := tD_s - sD_t$,

$$G(D_{st})s = s + t\frac{x}{1!} - s\frac{x^2}{2!} - t\frac{x^3}{3!} + \cdots = x\cos x + t\sin x,$$
$$G(D_{st})s^2 = (s\cos x + t\sin x)^2;$$
$$D_x(G(D_{st})s) = G(D_{st})t.$$

Multiplicative operator $G(D_{st})$ hold the following properties when D_{st} is any generalized operator of differentiation [19]:

$$G\sum_{k=1}^{n} \alpha_k f_k = \sum_{k=1}^{n} \alpha_k G f_k; \quad \alpha_k \in C, \tag{11}$$

$$Gf(s,t) = f(Gs, Gt); \quad (\text{when the function } f \text{ does not depend on } x), \tag{12}$$

$$G\frac{f_1(s,t)}{f_2(s,t)} = \frac{f_1(Gs,Gt)}{f_2(Gs,Gt)}. \tag{13}$$

For example, $G(s^k t^l) = (Gs)^k (Gt)^l$; $G(D_a)a^n = (x+a)^n$ when $D_a a^n := n a^{n-1}$; $G(D_a) f(a,s,t) = f(x+a,s,t)$. Readers are encouraged to check these simple equalities.

Also, the following equality holds:

$$(1 - L_a D_a) f(a,s,t) = f(0,s,t), \quad \text{when } L_a a^m = \frac{a^{m+1}}{m+1}; \quad m \in Z_0. \tag{14}$$

If $D_{ast} := D_a + P(s,t)D_s + Q(s,t)D_t$, then

$$G(D_{ast}) f(a) = G(D_a) f(a) = f(x+a), \tag{15}$$

when f depends on only one parameter a. In the general case,

$$G(D_{ast}) f(a,s,t) = f(G(D_{ast})a, G(D_{ast})s, G(D_{ast})t) = f(x+a, G(D_{ast})s, G(D_{ast})t). \tag{16}$$

2.4. Operator method for solving differential equations

Multiplicative operators can be used to construct exact analytical solutions of many differential equations [19]. It can be noted that we analyze general solutions only and do not investigate partial solutions. We will give few examples illustrating the multiplicative operator method.

2.4.1. Example 1

Let $P_1(x,s)$ and $P_2(x,s,t)$ be polynomials or ratios of polynomials of variables x, s and t. Then, the exact solution of a differential equation

$$y'_x = P_1(x,y); \quad y(a;s,t) = s; \quad a \in R \tag{17}$$

reads:

$$y = \sum_{k=0}^{+\infty} \frac{(x-a)^k}{k!}((D_a + P_1(a,s)D_s)^k s). \tag{18}$$

2.4.2. Example 2

Exact solution of differential equation

$$y''_{xx} = P_2(x,y,y'); \quad y(a;s,t) = s; \quad y'_x(x;s,t)\big|_{x=a} = t; \quad a \in R \tag{19}$$

reads:

$$y = \sum_{k=0}^{+\infty} \frac{(x-a)^k}{k!}((D_a + tD_s + P_2(a,s,t)D_t)^k s). \tag{20}$$

We will prove that Eq. (20) holds true.

Let D_{ast}: = $D_a + tD_s + P_2(a,s,t)D_t$ and $z(x;s,t,a) := G(D_{ast})s = \sum_{k=0}^{+\infty} \frac{x^k}{k!}(D_{ast}^k s)$. Then,

$$z'_x(x;s,t,a) = D_x G(D_{ast})s = D_{ast}G(D_{ast})s = G(D_{ast})D_{ast}s = G(D_{ast})t; \quad z''_{xx}(x;s,t,a) = D_x G(D_{ast})t = G(D_{ast})P_2(a,s,t)$$
$$= P_2(a+x, G(D_{ast})s, G(D_{ast})t) = P_2(a+x, z(x;s,t,a), z'_x(x;s,t,a)).$$

Thus, $z''_{xx}(x-a;s,t,a) = P_2(x, z(x-a;s,t,a), z'_x(x-a;s,t,a))$, and finally

$$y(x;s,t) = z(x-a;s,t,a),$$

what concludes the proof.

Similar techniques can be used to construct solutions of more complex differential equations (partial, nonlinear differential equations and their systems) [20]. These methods are based on the applicability of multiplicative operators for differential equations and can be considered as a generalization of the Picard method [19,20].

3. The rank of the Hankel matrix

We consider a sequence

$$p_0, p_1, \ldots := (p_j; j \in Z_0), \tag{21}$$

where the elements p_j, $j = 0,1,\ldots$ can be real, complex numbers, or even algebraic expressions comprising parameters $s,t,\ldots$. We construct a sequence of Hankel matrixes:

$$H_n = H_n(p_j; j \in Z_0) := \begin{bmatrix} p_0 & p_1 & \cdots & p_{n-1} \\ p_1 & p_2 & \cdots & p_n \\ & & \cdots & \\ p_{n-1} & p_n & \cdots & p_{2n-2} \end{bmatrix}, \quad n = 1,2,\ldots \tag{22}$$

Now we construct a sequence of determinants of the Hankel matrixes $(d_n; n \in N)$:

$$d_n := \det H_n. \tag{23}$$

Definition 1. The sequence $(p_j; j \in Z_0)$ has an H-rank $m \in Z_0$, $m < +\infty$

$$Hr(p_j; j \in Z_0) = m \tag{24}$$

if the sequence of determinants of the Hankel matrixes has the following structure:

$$(d_1, d_2, \ldots, d_m, 0, 0, \ldots), \tag{25}$$

where $d_m \neq 0$ and $d_{m+1} = d_{m+2} = \cdots = 0$.

For example, $Hr(j^2; j \in Z_0) = 3$ because the series of determinants is $(0,-1,-8,0,0,\ldots)$. Similarly, $Hr(\mu_1\rho_1^j + \mu_2\rho_2^j; j \in Z_0) = 2$ when $\mu_1, \mu_2 \neq 0$; $\rho_1 \neq \rho_2$ (0^0: = 1, but $0^2 = 0^3 = \cdots = 0$) and μ_1, μ_2, ρ_1, ρ_2 can depend on parameters $s,t,\ldots$. It is clear that $Hr(0,0,\ldots) = 0$, but a sequence $(j!; j \in Z_0)$ does not have an H-rank because all $d_n \neq 0$; $n \in N$.

Lets assume that a given sequence has an H-rank $Hr(p_j; j \in Z_0) = m$. Then a following characteristic equation can be constructed for that series [21]:

$$\begin{vmatrix} p_0 & p_1 & \cdots & p_m \\ p_1 & p_2 & \cdots & p_{m+1} \\ & & \cdots & \\ p_{m-1} & p_m & \cdots & p_{2m-1} \\ 1 & \rho & \cdots & \rho^m \end{vmatrix} = 0. \tag{26}$$

Expansion of the determinant in Eq. (26) yields an mth order algebraic equation ($d_m \neq 0$):

$$\rho^m + A_{m-1}\rho^{m-1} + \cdots + A_1\rho + A_0 = 0. \tag{27}$$

The following statement holds true [21]:

Let $Hr(p_j; j \in Z_0) = m$ and the multiplicity of roots $\rho_1, \rho_2, \ldots, \rho_l$ of the characteristic equation (Eq. (26)) is accordingly $m_1, m_2, \ldots, m_l$; $\sum_{r=1}^{l} m_r = m$. Then,

$$p_j = \sum_{r=1}^{l} \sum_{k=0}^{m_r-1} \mu_{rk} \binom{j}{k} \rho_r^{j-k}, \tag{28}$$

where $\mu_{rk}, \rho_r \in C$.

The opposite statement also holds true. If Eq. (28) holds, then

$$Hr(p_j; j \in Z_0) = m_1 + m_2 + \cdots + m_l. \tag{29}$$

Definition 2. A sequence is entitled as an algebraic sequence if elements of that sequence $(p_j; j \in Z_0)$ are defined by Eq. (28). We assume that $\mu_{rk} \binom{j}{k} \rho_r^{j-k} = 0$ if $\binom{j}{k} = 0$ what is true when $0 \leqslant j < k$.

In case when all roots of the characteristic equation are different, Eq. (28) obtains a more simple form:

$$p_j = \sum_{r=1}^{m} \mu_r \rho_r^j. \tag{30}$$

It can be noted that coefficients μ_{rk} (or just μ_r) can be found solving the linear algebraic system of equations ($\rho_1, \rho_2, \ldots, \rho_l$ are already determined):

$$\sum_{r=1}^{l} \sum_{k=0}^{m_r-1} \binom{j}{k} \rho_r^{j-k} \mu_{rk} = p_j; \quad j = 0, 1, \ldots, m-1. \tag{31}$$

The latter system of equations always has the only solution [21].

It can be also noted that one can use Eqs. (28) and (30) to calculate all elements of the series $(p_j; j \in Z_0)$ starting from p_{2m} if $Hr(p_j; j \in Z_0) = m$ and the first $2m$ elements of that series are known.

4. Expression of solutions of some differential equations in the form of sums of exponential functions and ratios of these sums

We will use a rather simple (though quite general) differential equation to illustrate our concept. Nevertheless, the same concept can be used for any other type of differential equation. We will discuss these questions in the last section.

4.1. Case when an H-rank exists

We will analyze the following initial problem:

$$y''_{xx} = P_2(y, y'); \quad y(a; s, t) = s; \quad y'_x(x; s, t)\big|_{x=a} = t. \tag{32}$$

Eq. (20) yields the solution of this problem:

$$y(x; s, t) = \sum_{j=0}^{+\infty} p_j(s, t) \frac{(x-a)^j}{j!}, \tag{33}$$

where

$$p_0(s, t) = s; \quad p_{j+1}(s, t) = D_{st} p_j(s, t) = D_{st}^{j+1} s, \tag{34}$$

and the generalized operator of differentiation is determined by:

$$D_{st} = t D_s + P_2(s, t) D_t. \tag{35}$$

The generalized operator D_{st} does not depend on parameter a because $P_2(y, y')$ does not depend on x.

Lets assume that the produced series $(p_j(s, t); j \in Z_0)$ has an H-rank equal to m. Then the characteristic equation (Eq. (26)) has m roots $\rho_k(s, t)$, $k = 1, 2, \ldots, m$.

If all roots are different:

$$\rho_k(s, t) \neq \rho_l(s, t) \quad \text{for } k \neq l. \tag{36}$$

Eq. (30) yields:

$$y(x; s, t) = \sum_{j=0}^{+\infty} \frac{(x-a)^j}{j!} (\mu_1(s, t) \rho_1^j(s, t) + \cdots + \mu_m(s, t) \rho_m^j(s, t)) = \sum_{r=1}^{m} \mu_r(s, t) \exp((x-a)\rho_r(s, t)). \tag{37}$$

In other words, Eqs. (24) and (36) produce necessary and sufficient conditions when the solution of the differential equation (32) can be expressed in a finite sum (comprising m terms) of exponential functions.

If Eq. (36) does not hold (some of roots of the characteristic equation are multiple), the structure of the solution is little bit more complex [21]:

$$y(x;s,t) = \sum_{r=1}^{l} Q_r(x;s,t)\exp((x-a)\rho_r(s,t)), \tag{38}$$

where $Q_r(x;s,t)$, $r = 1,2,\dots,l$; $l < m$ are polynomials of x and depend on initial conditions s and t.

4.2. Case when an H-rank does not exist

Let the series of coefficients $(p_j(s,t); j \in Z_0)$ of the differential equation (32) does not have an H-rank.

Then it is useful to follow the He's method [1] and to change the variable:

$$\exp(x) := z; \quad x = \ln(z); \quad x \in R; \quad z > 0. \tag{39}$$

Then, introduction of a new parameter ω yields:

$$\omega = \omega(z) = y(\ln z); \quad \omega'_z = \frac{1}{z} y'_x(x)\big|_{x=\ln z}; \quad \omega''_{zz} = \frac{1}{z^2}(y''_{xx}(x) - y'_x(x))\big|_{x=\ln z}, \tag{40}$$

or

$$y'_x(x)\big|_{x=\ln z} = z \cdot \omega'_z; \quad y''_{xx}(x)\big|_{x=\ln z} = z^2\omega''_{zz} + z\omega'_z. \tag{41}$$

Thus, the initial problem in Eq. (32) reads:

$$\omega''_{zz} = \frac{1}{z^2}(P_2(\omega, z\omega'_z) - z\omega'_z); \quad \omega(\exp(a);s,t) = s; \quad \omega'_z(z;s,t)\big|_{z=\exp(a)} = t. \tag{42}$$

We will entitle Eq. (42) as the image of the differential equation (32). Let

$$\hat{P}_2(z,\omega,\omega'_z) := \frac{1}{z^2}(P_2(\omega,\omega'_z) - z\omega'_z). \tag{43}$$

Then,

$$\omega(z;s,t,\exp(a)) = \sum_{j=0}^{+\infty} \frac{(z-\exp(a))^j}{j!}\hat{p}_j(s,t,\exp(a)), \tag{44}$$

where $\hat{p}_j(s,t,a) = (D_a + tD_s + \hat{P}_2(a,s,t)D_t)^j s$.

Let's assume that

$$Hr\left(\frac{\hat{p}_j(s,t,a)}{j!}; j \in Z_0\right) = m. \tag{45}$$

Then, if all roots $\hat{\rho}_r(s,t,\exp(a))$; $r = 1,2,\dots,m$ of the characteristic equation of the sequence $\hat{p}_j$ (corresponding to Eq. (42)) are different,

$$\begin{aligned}\omega(z;s,t) &= \sum_{j=0}^{+\infty}(z-\exp(a))^j \sum_{r=1}^{m} \mu_r(s,t,\exp(a))\hat{\rho}_r^j(s,t,\exp(a)) \\ &= \sum_{r=1}^{m}\mu_r(s,t,\exp(a))\sum_{j=0}^{+\infty}(\hat{\rho}_r(s,t,\exp(a))(z-\exp(a)))^j = \sum_{r=1}^{m}\frac{\mu_r(s,t,\exp(a))}{1-\hat{\rho}_r(s,t,\exp(a))(z-\exp(a))}.\end{aligned} \tag{46}$$

Thus,

$$\omega(z;s,t) = \frac{\sum_{l=1}^{n_1} a_l z^l}{\sum_{l=1}^{n_2} b_l z^l}. \tag{47}$$

Inverse change of variables ($z = \exp(x)$) produces:

$$y(x;s,t) = \frac{\sum_{l=1}^{n_1} a_l \exp(lx)}{\sum_{l=1}^{n_2} b_l \exp(lx)}. \tag{48}$$

If some of the roots of the characteristic equation are multiple, relationships become little more complex, but the solution can be also expressed in a form comprising ratio of sums of exponential functions.

4.3. Examples

In this section, we will give three simple examples illustrating the flow of computations. Differential equations in all these examples can be solved by other classical methods what helps to check the validity of the proposed concept. More complex differential equations would require symbolic computations. As mentioned previously, the object of this paper is just to present a generalized algorithm for the discussed concept.

4.3.1. Example 1– H-rank exists for the original equation

We will illustrate the concept using a classical second order homogenous differential equation with constant coefficients:

$$y'' + ay' + by = 0; \quad a, b \in R; \quad y(0; s, t) = s; \quad y'_x(x; s, t)\big|_{x=0} = t. \tag{49}$$

Then, $y'' = -ay' - by$; therefore $P_2(s,t) = -(bs + at)$ and $p_j(s,t) = (tD_s - (bs + at)D_t)^j s, j = 0, 1, 2, \ldots$

The sequence of determinants of the Henkel matrixes constructed from $(p_j(s,t); j \in Z_0)$ reads:

$$(s, -(bs^2 + ast + t^2), 0, 0, \ldots). \tag{50}$$

Thus $Hr(p_j(s,t); j \in Z_0) = 2$ if only $bs^2 + ast + t^2 \neq 0$. Then the characteristic equation takes the form:

$$\begin{vmatrix} s & t & -(bs+at) \\ t & -(bs+at) & abs + a^2t - bt \\ 1 & \rho & \rho^2 \end{vmatrix} = 0. \tag{51}$$

Expansion of the determinant and elementary simplifications help to determine the roots of the characteristic equation: $\rho_{1,2} = \frac{1}{2}(-a \pm \sqrt{a^2 - 4b})$. Let $a^2 - 4b \neq 0$. Then $\rho_1 \neq \rho_2$ and solving the linear system of Eq. (31) produces:

$$p_j(s,t) = \frac{t - \rho_2 s}{\sqrt{a^2 - 4b}} \rho_1^j - \frac{t - \rho_1 s}{\sqrt{a^2 - 4b}} \rho_2^j. \tag{52}$$

Thus

$$y(x; s, t) = \frac{t - \rho_2 s}{\sqrt{a^2 - 4b}} \exp(\rho_1 x) - \frac{t - \rho_1 s}{\sqrt{a^2 - 4b}} \exp(\rho_2 x). \tag{53}$$

Eq. (53) is a well known solution of differential equation (49).

Special solutions are produced when $bs^2 + ast + t^2 = 0$ or $\rho_1 = \rho_2$, but we omit the details for the brevity.

4.3.2. Example 2– H-rank does not exist for the original equation but exists for the image equation

We consider a first order differential equation:

$$y' = 1 - y^2; \quad y(0; s) = s. \tag{54}$$

Coefficients of this differential equation $(((1 - s^2)D_s)^j s; j \in Z_0)$ do not have an H-rank. The image equation reads:

$$\omega'_z = \frac{1}{z}(1 - \omega^2); \quad \omega(1; s) = s. \tag{55}$$

The coefficients of the image equation $\hat{p}_j(s,a) = \left(\frac{1}{a}(1 - s^2)D_s\right)^j s;\ j \in\ Z_0$ have an H-rank; $Hr\left(\frac{\hat{p}_j(s,a)}{j!}; j \in Z_0\right) = 3$. Roots of the characteristic algebraic equation (27) (at $m = 3$) are:

$$\hat{\rho}_1 = 0; \quad \hat{\rho}_2 = \frac{-a + ic}{a^2 + c^2}; \quad \hat{\rho}_3 = \frac{-a - ic}{a^2 + c^2}, \tag{56}$$

where $c = a\sqrt{\frac{1-s}{1+s}}$, and coefficients μ_1, μ_2 and μ_3 read:

$$\mu_1 = 1; \quad \mu_2 = -\frac{c(c + ia)}{a^2 + c^2}; \quad \mu_3 = -\frac{c(c - ia)}{a^2 + c^2}. \tag{57}$$

Thus, $\hat{p}_0(s,a) = s$; $\frac{\hat{p}_j(s,a)}{j!} = \mu_2 \hat{\rho}_2^j + \mu_3 \hat{\rho}_3^j$; as $\hat{\rho}_1^0 = 0^0 = 1$; $\hat{\rho}_1^j = 0, j = 1, 2, \ldots$. Elementary mathematical transformations yield:

$$\sum_{j=0}^{+\infty} \frac{\hat{p}(s,a)}{j!}(z - a)^j = \frac{(1+s)z^2 - a^2(1-s)}{(1+s)z^2 + a^2(1-s)}. \tag{58}$$

The solution of the differential equation (55) takes the following form (note that $a = 1$):

$$\omega(z; s) = \frac{(1+s)z^2 - (1-s)}{(1+s)z^2 + (1-s)}. \tag{59}$$

Then, finally, the solution of the original differential equation (54) reads:

$$y(x; s) = \frac{(1+s)\exp(2x) - (1-s)}{(1+s)\exp(2x) + (1-s)} = \frac{(1+s)\exp x - (1-s)\exp(-x)}{(1+s)\exp x + (1-s)\exp(-x)}. \tag{60}$$

Clearly, this solution can be easily determined by a simple separation of variables in the original equation.

4.3.3. Example 3

Now we consider

$$y' = -y^2; \quad y(0;s) = s. \tag{61}$$

The series of coefficients of this differential equation $\left((-s^2D_s)^j s; j \in Z_0\right) = \left((-1)^j s^{j+1} j!; j \in Z_0\right)$ does not have an H-rank. The image differential equation is:

$$\omega'_z = \frac{-\omega^2}{z}; \quad \omega(1,s) = s. \tag{62}$$

Its sequence of coefficients $\left(\frac{1}{j!}\left(\frac{s^2}{a}D_s\right)^j s; j \in Z_0\right)$ also does not have an H-rank. Thus, the solution of Eq. (61) cannot be expressed in a ratio of sums of exponential functions.

5. Discussion and concluding remarks

We have constructed an analytical criterion determining if a solution of a differential equation can be expressed in an analytical form comprising exponential functions.

Moreover, the employment of this criterion does not only give an answer to the above-stated question. It gives the structure of the solution. One does not have to guess what the form of the solution is.

It can be noted that all computations are straightforward when the structure of the solution is identified (in fact one needs to solve linear algebraic equations). The load of symbolic calculations is brought before the structure of the solution is identified. This is in contrary to the Exp-function method where the structure of the solution is first guessed, and then symbolic calculations are exploited for the identification of parameters.

The flow of calculations can be described by following steps. Given an initial problem (an ordinary differential equation and initial conditions), one must use the operator method to construct the solution in the form of a series $y(x;s,t) = \sum_{j=0}^{+\infty} p_j(s,t)\frac{(x-a)^j}{j!}$. Then, the H-rank for the sequence of coefficients $(p_j(s,t); j \in Z_0)$ must be calculated. If the H-rank exists and is equal to m, one has to find roots $\rho_k(s,t)$, $k = 1,2,\ldots,m$ of the characteristic equation. The exact solution is expressed in a finite sum of exponential functions.

If the H-rank does not exist, one has to change the variable $\exp(x) = z$ and construct the image differential equation. Then, the operator method must be used to construct the solution in the form of a series $\omega(z;s,t,a) = \sum_{j=0}^{+\infty}\frac{(z-\exp(a))^j}{j!}\hat{p}_j(s,t,\exp(a))$. Then the H-rank for the sequence $\left(\frac{\hat{p}_j(s,t,a)}{j!}; j \in Z_0\right)$ must be calculated. If the H-rank exists and it equal to m, one has to find roots $\hat{\rho}_r(s,t,a)$; $r = 1,2,\ldots,m$ of the characteristic equation. The exact solution is expressed in a ratio of finite sums of exponential functions.

If the H-rank does not exist for the image equation also, the exact solution cannot be expressed in a form comprising only exponential functions.

So far we have been dealing with ordinary differential equations (any order). The operator method for the solution of differential equations can be directly used for the construction of the solution in the form of a series [18]. In this sense, our method is more general compared to the Exp-function method.

Anyway, when solitary solutions are considered only, partial differential equations are usually transformed to ordinary autonomous differential equations by a special variable change [1–17]. Therefore, we concentrated only on ordinary differential equations in this manuscript. Direct application of our concept to partial differential equations is a subject of future research.

From the other point of view, the Exp-function method can be considered as technique employing a variable change $\exp(x) = z$. Clearly, one can use different variable changes. A typical example is a tanh-method for solving nonlinear differential equations [22,23]. The main idea is still the same from the point of view of our proposed method. If a sequence of coefficients has an H-rank, the series can be expressed in a finite algebraic expression.

We can easily adapt our method to any other variable changes. In principle, the structure of the algorithm remains the same. One has to construct the image differential equation if the H-rank does not exist for the original equation. We will illustrate the concept for a general variable change $z = \varphi(x) = \sum_{n=0}^{+\infty}\frac{x^n}{q_n}$, where q_n are coefficients of the expansion. Then, the solution of the differential equation can be expressed in a finite sum comprising powers of $\varphi(x)$ if only $Hr\left(\frac{p_j q_j}{j!}; j \in Z_0\right) = m$; where p_j are coefficients of the solution series produced by the operator method: $y(x) = \sum_{j=0}^{+\infty} p_j\frac{(x-a)^j}{j!}$. We do not prove this statement for the sake of the brevity and leave it for the future research.

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