A segmenting scheme for evaluating exact high-order modes of uniform Timoshenko beams

Wei Xu [a], Maosen Cao [a,*], Keqin Ding [b], Minvydas Ragulskis [c], Xiang Zhu [a]

[a] *Department of Engineering Mechanics, Hohai University, Nanjing 210098, PR China*
[b] *Structural Health Monitoring Department, China Special Equipment Inspection and Research Institute, Beijing 100013, PR China*
[c] *Center for Nonlinear Systems, Kaunas University of Technology, Kaunas LT-51368, Lithuania*

Keywords:
Uniform Timoshenko beam
Exact high-order mode
Round-off error
Segmenting scheme
Modal frequency
Exact dynamic stiffness method

ABSTRACT

Numerical evaluation of exact high-order modes of uniform Timoshenko beams is a classic problem in the field of acoustics and vibration. In the classic method, however, the determinant of the frequency matrix dominated by hyperbolic functions can increase exponentially and quickly reach the precision limitation of a computer. As a result, a bottleneck occurs in the evaluation of high-order modes due to round-off errors. Addressing this bottleneck, this study proposes an enhanced classic method relying on a segmenting scheme for evaluating exact high-order modes of uniform Timoshenko beams. With the segmenting scheme, a uniform Timoshenko beam is uniformly segmented into several segments, whereby the hyperbolic functions involved in the determinant of the frequency matrix become much smaller for the same given frequency. Accordingly, under the fixed precision limitation in a computer, higher-order modes can be obtained. The capacity of this enhanced classic method for evaluating exact high-order modes is validated by scenarios of a uniform Timoshenko beam with different numbers of segments. The results show that the high-order modal frequencies and mode shapes can be properly obtained. The accuracy of the modal frequencies is verified by the well-established exact dynamic stiffness method with the Wittrick-Williams algorithm. The results show that the modal frequencies are highly accurate. Applied to beams of different materials, the artificial segmenting is more suitable than the natural segmenting in evaluating high-order modes.

1. Introduction

Investigating high-frequency vibration behaviors of beam-type structural components is crucial for mechanical and aerospace structures [1–3]. As the vibration response can be linearly decomposed into mode shapes, numerical evaluation of high-order modes of a beam is a fundamental issue for the high-frequency vibration response.

The classic method of solving exact modes of beams can be found in vibration textbooks [4]: modal frequencies can be determined by solving the frequency equation, and the corresponding mode shape can be easily obtained by substituting solved modal frequency into the mode shape function. The classic method permits only the first 12 or so modes when solving the frequency equation [5,6], as the determinant of the frequency matrix behaves unreasonably for frequencies higher than the 12th modal frequency, termed the 12th-mode bottleneck [7]. The bottleneck in high-order modes renders the classic method incomplete and has become a challenging problem that has attracted the attentions of many researchers over the past three decades. Tang [5] attributed this bottleneck problem to the round-off error in the floating-point math caused by the precision limitation of the floating-point representation by a digital computer. Therefore, expanding the valid frequency range is the key to breaking through the bottleneck in the classic method, by which exact high-order modes can be properly evaluated. Addressing this problem, two types of methods have been developed, called the direct method and the indirect method. In the direct method, high-order modes can be obtained by reformulating the expression for the mode shape functions of Euler–Bernoulli beams, whereby high-order modes can be approximately evaluated [5,6]. Recently, Goncalves et al. [8] reformulated exact expressions of Euler–Bernoulli beam mode shape functions, which are more numerically stable than approximate expressions. The indirect method commonly utilizes the exact dynamic stiffness method (EDSM) to determine any modal frequency by counting the number of modal frequencies below a given frequency using the well-known Wittrick-Williams (W-W)

* Corresponding author.
E-mail address: cmszhy@hhu.edu.cn (M. Cao).

algorithm [9–16], which can solve all modal frequencies of a structure in theory.

However, both direct and indirect method have limitations compared to the classic method. The direct method is only used in Euler–Bernoulli beams so far, which has a noticeable limitation in a physical sense. The shear effect on deformations of an Euler–Bernoulli beam is ignored; thus, the solution to high-order modes of the beam is inaccurate [17]. Instead of the Euler–Bernoulli beam theory, the Timoshenko beam theory should be employed to depict the vibration of beams in high-order modes. Nevertheless, it would be difficult to reformulate the mode shape functions of Timoshenko beams due to their complexity. In contrast to the classic method that can find all the successive modal frequencies in the range of interest, the indirect method finds one modal frequency at a time. With these concerns, this study proposes a segmenting scheme and integrates it into the classic method, whereby an enhanced classic method is proposed. It can break through the bottleneck in high-order modes, and therefore exact high-order modes of uniform Timoshenko beams can be evaluated.

The rest of this paper is organized as follows. Section 2 introduces the bottleneck of the classic method in the numerical evaluation of exact high-order modal frequencies of uniform Timoshenko beams. Section 3 proposes the segmenting scheme to enhance the classic method, whereby the enhanced classic method is developed. Section 4 verifies the capacity of the enhanced classic method in achieving exact high-order modes of uniform Timoshenko beams, and demonstrates its superiority to the classic method. The accuracy of the evaluated modal frequencies is validated by the well-established EDSM with the W-W algorithm. Section 5 presents the conclusions of this study.

2. Bottleneck of classic method in high-order modes

2.1. Numerical evaluation of modes using classic method

For the free transverse vibration of a uniform Timoshenko beam [18], the abscissa x along the beam of length L is set under a global coordinate system. By introducing the dimensionless abscissa $\zeta = \frac{x}{L}$, the governing differential equations for the transverse deflection W and the slope due to bending ψ in spatial domain can be established [19]:

$$\frac{d^4}{d\zeta^4}W(\zeta) + 2\alpha\frac{d^2}{d\zeta^2}W(\zeta) + \beta W(\zeta) = 0, \tag{1a}$$

$$\frac{d^4}{d\zeta^4}\psi(\zeta) + 2\alpha\frac{d^2}{d\zeta^2}\psi(\zeta) + \beta\psi(\zeta) = 0, \tag{1b}$$

where $\alpha = \frac{\tau(r+s)}{2}$, $\beta = \tau(\tau rs - 1)$ are functions of $r = \frac{I}{AL^2}$, $\vartheta = \frac{E}{\kappa G}$, $s = \vartheta r$, and $\tau = \frac{\rho A}{EI}L^4\omega^2$, with ω, E, G, ρ, I, A, and κ the angular frequency, elastic modulus, shear modulus, density, moment of inertia, cross-sectional area and shear coefficient, respectively. The solutions to Eqs. (1) can be expressed as [20–22]

$$W(\zeta) = C_1\cosh\gamma_1\zeta + C_2\sinh\gamma_1\zeta + C_3\cos\gamma_2\zeta + C_4\sin\gamma_2\zeta, \tag{2a}$$

$$\psi(\zeta) = C_1m_1\sinh\gamma_1\zeta + C_2m_1\cosh\gamma_1\zeta + C_3m_2\sin\gamma_2\zeta - C_4m_2\cos\gamma_2\zeta, \tag{2b}$$

where $\gamma_1 = \left(\sqrt{\alpha^2-\beta}-\alpha\right)^{1/2}$, $\gamma_2 = \left(\sqrt{\alpha^2-\beta}+\alpha\right)^{1/2}$, $m_1 = \frac{\tau s+\gamma_1^2}{\gamma_1}$, $m_2 = \frac{\tau s-\gamma_2^2}{\gamma_2}$, and C_1 to C_4 are constants to be determined. Considering boundary conditions of the displacement, slope due to bending, bending moment, and shear force [20–22] at two ends $\zeta = 0$ or $\zeta = 1$ yields a set of homogeneous linear equations in terms of the vector $C = (C_1, C_2, C_3, C_4)^T$:

$$\mathbf{D}(\omega)\mathbf{C} = 0. \tag{3}$$

By vanishing the determinant of the frequency matrix $\mathbf{D}(\omega)$, denoted as $|\mathbf{D}(\omega)|$, a series of modal frequencies ω_j can be produced. As the $|\mathbf{D}(\omega)|$ can be extremely large for high frequencies, to clearly illustrate the full magnitudes of the profile, its inverse hyperbolic sine function $Y(|\mathbf{D}(\omega)|)$ is utilized instead of $|\mathbf{D}(\omega)|$:

$$Y(|\mathbf{D}(\omega)|) = \sinh^{-1}(|\mathbf{D}(\omega)|) = \ln\left(|\mathbf{D}(\omega)| + \sqrt{|\mathbf{D}(\omega)|^2 + 1}\right). \tag{4}$$

2.2. Bottleneck in high-order modes

To evaluate exact modal frequencies of a uniform Timoshenko beam using the classic method, a cantilever beam is considered for demonstration since it is the most representative beam type to study the bottleneck in high-order modes [5,7]. Assuming the cross-sectional area 1e−5 m^2, moment of inertia 8.33e−13 m^4, length 1.2 m, elastic modulus 7e4 MPa, shear modulus 2.7e4 MPa, shear coefficient 0.85, and density 2700 kg/m^3, the profile of $Y(|\mathbf{D}(\omega)|)$ versus frequency ω is obtained and behaves unreasonably after the 12th modal frequency (marked with a red circle), provoking the bottleneck in high-order modes as illustrated in Fig. 1. The cause of the bottleneck in high-order modes lies in the conflict between the unconstrained increase of the determinant and the precision limitation of the floating-point representation in a computer [23]. When evaluating values of $|\mathbf{D}(\omega)|$, the hyperbolic functions $\cosh\gamma_1\zeta$ and $\sinh\gamma_1\zeta$ ($\zeta = 1$) increase exponentially with ω [24,25] and quickly reach the limitation of digital representation. As a result, round-off errors in the floating-point math occur when ω exceeds an upper bound. It should be noted that simply-supported Timoshenko beams do not suffer from such bottlenecks in high-order modes because their terms of $\cosh\gamma_1\zeta$ and $\sinh\gamma_1\zeta$ vanish. In contrast, hyperbolic functions always exist in the frequency determinant for a Timoshenko beam with other types of boundary conditions.

3. Enhanced classic method relying on a segmenting scheme

A segmenting scheme is integrated into the classic method to enhance its capability of numerically evaluating exact high-order modes. A uniform Timoshenko beam in Fig. 2 is uniformly segmented into n segments, each with the length $S_i = \frac{L}{n}$, where joint cross-sections of two adjacent segments are indicated by red dashed lines. The distance from the origin to the i-th joint cross-

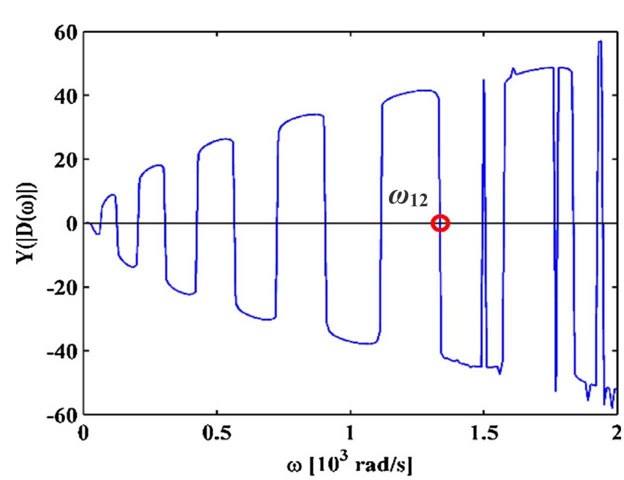

Fig. 1. $Y(|\mathbf{D}(\omega)|)$ for uniform cantilever Timoshenko beam.

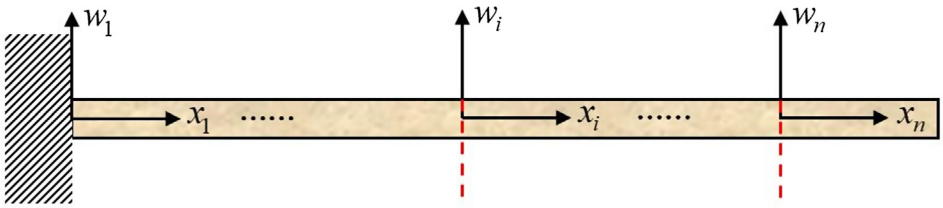

Fig. 2. Uniform Timoshenko beam segmented into n segments with n local coordinate systems.

section is denoted as L_i, and the origin is set at $L_0 = 0$. The i-th $(i = 1, \ldots, n)$ local coordinate system (x_i, w_i) is introduced into the i-th beam segment to individually portray its deformation with the origin fixed at the left end. x_i in the i-th local coordinate system is related to x in the global coordinate system by $x_i = x - L_{i-1}$, and can be represented in its dimensionless form $\eta_i = x_i/L$.

As the deformation of every point in a beam is identical under the global coordinate system and local coordinate systems, the corresponding spatial solutions must be identical as well:

$$W(\zeta) = W_i(\eta_i), \psi(\zeta) = \psi_i(\eta_i) \tag{5}$$

where W_i and ψ_i are the transverse deflection and the slope due to bending for the i-th beam segment under the i-th local coordinate system, respectively.

Substituting Eq. (5) into Eqs. (1) together with $x_i = x - L_{i-1}$, we have the new governing differential equations under local coordinate systems:

$$\frac{d^4}{d\eta_i^{\,4}} W_i(\eta_i) + 2\alpha \frac{d^2}{d\eta_i^{\,2}} W_i(\eta_i) + \beta W_i(\eta_i) = 0, \tag{6a}$$

$$\frac{d^4}{d\eta_i^{\,4}} \psi_i(\eta_i) + 2\alpha \frac{d^2}{d\eta_i^{\,2}} \psi_i(\eta_i) + \beta \psi_i(\eta_i) = 0. \tag{6b}$$

The solutions to Eqs. (6) can be expressed as

$$W_i(\eta_i) = C_{i,1} \cosh\gamma_1\eta_i + C_{i,2} \sinh\gamma_1\eta_i + C_{i,3} \cos\gamma_2\eta_i + C_{i,4} \sin\gamma_2\eta_i, \tag{7a}$$

$$\psi_i(\eta_i) = C_{i,1} m_1 \sinh\gamma_1\eta_i + C_{i,2} m_1 \cosh\gamma_1\eta_i + C_{i,3} m_2 \sin\gamma_2\eta_i - C_{i,4} m_2 \cos\gamma_2\eta_i, \tag{7b}$$

which are general for uniform Timoshenko beams with arbitrary geometrical and material parameters. $C_{i,1}$ to $C_{i,4}$ are constants to be determined by boundary conditions together with continuity conditions of displacement, slope, bending moment, and shear force at all joint cross-sections of adjacent beam segments.

Similar to Eq. (4), modal frequencies can be evaluated by vanishing $Y(|\mathbf{D}^*(\omega)|)$, where superscript $*$ denotes use of the segmenting scheme.

In contrast to the determinant of the frequency matrix $|\mathbf{D}(\omega)|$ in the classic method, $|\mathbf{D}^*(\omega)|$ in the enhanced classic method is beneficial for numerically evaluating exact high-order modes. To be specific, in the classic method, the hyperbolic functions $\cosh\gamma_1\zeta$ and $\sinh\gamma_1\zeta$ $(\zeta = 1)$ in $|\mathbf{D}(\omega)|$ dominate the upper bound of valid ω; in contrast, in the enhanced classic method, the hyperbolic functions $\cosh\gamma_1\eta_i$ and $\sinh\gamma_1\eta_i$ $(\eta_i = l_i = \frac{1}{n})$ in $|\mathbf{D}^*(\omega)|$ dominate the upper bound of valid ω. Due to the fact that $\eta_i = \frac{1}{n} < \zeta = 1$, under the fixed precision limitation in a computer, the γ_1 and γ_2 in $|\mathbf{D}^*(\omega)|$ are allowed to be larger than those in $|\mathbf{D}(\omega)|$, resulting in a larger upper bound of valid ω. Thus, by virtue of the larger upper bound of valid ω permitted by the segmenting scheme, the bottleneck in high-order modes can be excluded from the frequency range of interest, such that high-order modal frequencies can be properly obtained. Accordingly, the high-order mode shapes can be obtained.

4. Capacity validation

To validate the capacity of the enhanced classic method in achieving exact high-order modes of uniform Timoshenko beams, the same cantilever beam in Section 2.2 is considered for comparison of the classic method and the enhanced classic method. Scenarios for the enhanced classic method with two, three, and four segments are considered, respectively. For the scenario with two segments, the profile of $Y(|\mathbf{D}^*(\omega)|)$ versus frequency ω in Fig. 3 clearly shows that it breaks through the 12th-mode bottleneck:

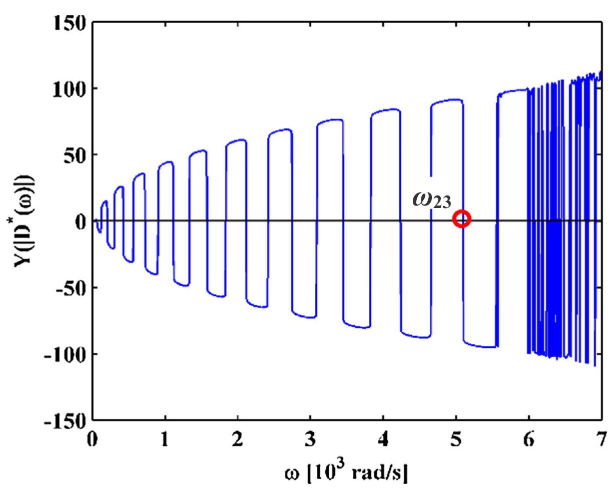

Fig. 3. $Y(|\mathbf{D}^*(\omega)|)$ for uniform cantilever Timoshenko beam with two segments.

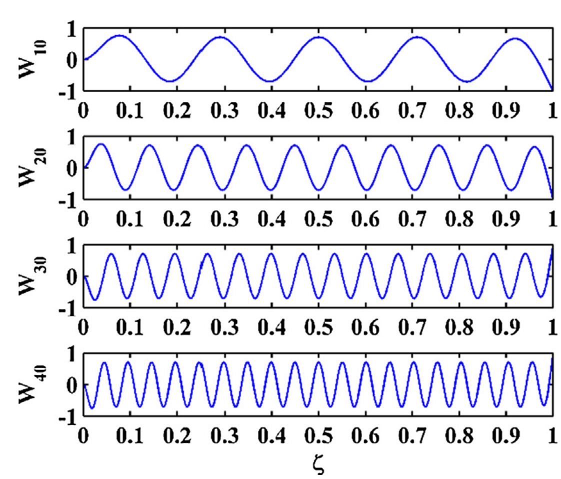

Fig. 4. High-order mode shapes of uniform cantilever Timoshenko beam.

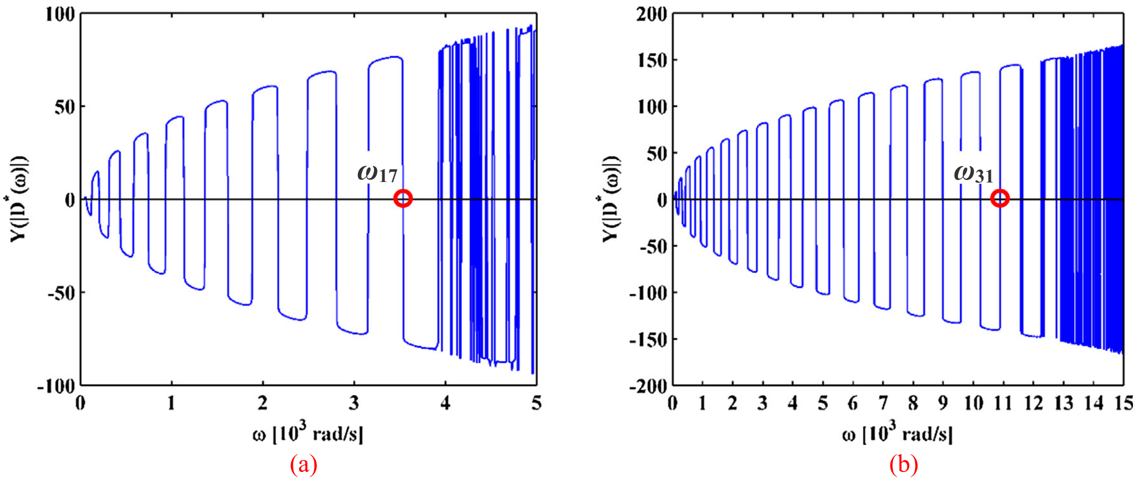

Fig. 5. $Y(|\mathbf{D}^*(\omega)|)$ for Timoshenko beam of different materials with (a) natural (b) artificial segmenting.

modal frequencies up to the 23th (marked with a red circle) can be properly identified, far beyond the first 12 modal frequencies (Fig. 1) by the classic method. To explore higher modal frequencies, the beam is further segmented into more segments. Correspondingly, the first 33 and 44 modal frequencies can be identified with three and four segments, respectively. It can be seen from the results, the permitted mode number seems to be about 11 times of the number of discretization, i.e., the number of beam segments which is less than five. Much higher modal frequencies can be evaluated if the beam is segmented into more segments. Thus, the segmenting scheme can effectively improve the classic method by circumventing the bottleneck in high-order modes.

The accuracy of the modal frequencies determined by the enhanced classic method are verified by the well-established EDSM for Timoshenko beams [26] with the W-W algorithm [10]. Both the enhanced classic method and the EDSM can produce exact modes because they are based on the same exact mode shape expression for discrete Timoshenko beam segments, and the frequency matrix in the enhanced classic method can be regarded as a form of a dynamic stiffness matrix. Taking this into account, the EDSM is very suitable to evaluate the accuracy of the enhanced classic method, rather than other approximate numerical methods like the finite element (FE) method whose accuracy depends on meshing and shape functions, although they are more popular for engineers. For scenarios with two, three, and four segments, the first 33 modal frequencies evaluated by the two methods are almost identical. Most of the relative errors are less than 0.001%, whose maximum relative errors 0.001065%, 0.06388%, and 0.0003964% occur at the 23th, 8th, and 40th modes for scenarios with two, three, and four segments, respectively. Such small errors can be attributed to the extensive use of matrix manipulations in a computer [27].

By submitting the determined modal frequencies into Eq. 7(a), the corresponding mode shapes of the uniform Timoshenko beam can be easily obtained. Using the four-segment Timoshenko beam model, the 10th, 20th, 30th, and 40th mode shapes (normalized with unit amplitudes) at 909.1018, 3829.0002, 8758.1173, and 15689.5914 rad/s are obtained and shown in Fig. 4, respectively.

To explore the wider application of the enhanced classic method, a straight collinear cantilever beam of two different materials is considered. The elastic modulus and density in $\zeta \in [0, 1/3]$ are 1.1 and 1.2 times of those given in Section 2.2, respectively; and are 0.9 and 0.8 times in $\zeta \in [1/3, 1]$, respectively. When the beam is naturally segmented into two segments by different materials, only the first 17 modal frequencies are permitted, as shown in Fig. 5(a); in contrast, by further artificially dividing the beam into three uniform segments, the first 31 modal frequencies can be properly identified, as shown in Fig. 5(b). Thus, the artificial segmenting is more suitable than the natural segmenting in evaluating high-order modes for beams of different materials.

5. Conclusions

Numerical evaluation of exact high-order modes of beams using the classic method is a classic problem in the field of acoustics and vibration, but bottlenecks in high-order modes render the classic method incomplete. With this concern, this study proposed a segmenting scheme for the classic method, whereby an enhanced classic method is developed to break through the bottleneck and obtain exact high-order modes. The following conclusions are drawn.

(1) In the classic method, the cause of the bottleneck in high-order modes lies in the conflict between the unconstrained increase in the determinant of a frequency matrix and the limited precision of the floating-point representation for a computer. Round-off errors in the floating-point math occur when the frequency exceeds an upper bound.
(2) In the enhanced classic method, a uniform Timoshenko beam is uniformly segmented into several segments, whereby the hyperbolic functions involved in the frequency matrix become smaller for a given frequency. Under the fixed precision limitation in a computer, higher-order modes are available.
(3) The enhanced classic method has a wide application including beams of different materials. The artificial segmenting can obtain higher modes than the natural segmenting by different materials.

Acknowledgments

This work is supported by the National Key R&D Program of China (No. 2018YFF0214705) and the National Natural Science Foundation of China (Nos. 51708173 and 11772115).

References

[1] Bernhard RJ, Huff JE. Structural-acoustic design at high frequency using the energy finite element method. J Vib Acoust 1999;121:295–301.

[2] Cao M, Ostachowicz W, Li G, Xu W, Ashoory AR. Identification of incipient damage using high-frequency vibrational responses. Shock Vib 2015:153085.
[3] Abusoua A, Daqaq MF. On using a strong high-frequency excitation for parametric identification of nonlinear systems. J Vib Acoust 2017;139:051012.
[4] Leissa AW, Qatu MS. Vibration of Continuous Systems. McGraw Hill; 2011.
[5] Tang Y. Numerical evaluation of uniform beam modes. J Eng Mech 2003;129:1475–7.
[6] Goncalves PJP, Brennan MJ, Elliott SJ. Numerical evaluation of high-order modes of vibration in uniform Euler-Bernoulli beams. J Sound Vib 2007;301:1035–9.
[7] Xu W, Cao M, Ren Q, Su Z. Numerical evaluation of high-order modes for stepped beam. J Vib Acoust 2014;136:014503.
[8] Goncalves PJP, Peplow A, Brennan MJ. Exact expressions for numerical evaluation of high order modes of vibration in uniform Euler-Bernoulli beams. Appl Acoust 2018;141:371–3.
[9] Williams FW, Wittrick WH. An automatic computational procedure for calculating modal frequencies of skeletal structures. Int J Mech Sci 1970;12:781–91.
[10] Howson WP, Williams FW. Natural frequencies of frames with axially loaded Timoshenko Members. J Sound Vib 1973;26:503–15.
[11] Williams FW, Wittrick WH. Exact buckling and frequency calculations surveyed. J Struct Eng 1983;109:169–87.
[12] Banerjee JR. Dynamic stiffness formulation for structural elements: A general approach. Comput Struct 1997;63:101–3.
[13] Banerjee JR. Dynamic stiffness formulation and free vibration analysis of centrifugally stiffened Timoshenko beams. J Sound Vib 2001;247:97–115.
[14] Banerjee JR. Free vibration of sandwich beams using the dynamic stiffness method. Comput Struct 2003;81:1915–22.
[15] Howson WP, Zare A. Exact dynamic stiffness matrix for flexural vibration of three-layered sandwich beams. J Sound Vib 2005;282:753–67.
[16] Banerjee JR, Cheung CW, Morishima R, Perera M, Njuguna J. Free vibration of a three-layered sandwich beam using the dynamic stiffness method and experiment. Int J Solids Struct 2007;44:7543–63.
[17] Han SM, Benaroya H, Wei T. Dynamics of transversely vibrating beams using four engineering theories. J Sound Vib 1999;225:935–88.
[18] Timoshenko S. On the transverse vibrations of bars of uniform cross-section. Phil Mag 1922;43:125–31.
[19] Huang TC. The effect of rotatory inertia and of shear deformation on the frequency and normal mode equations of uniform beams with simple end conditions. J Appl Mech 1961;28:579–84.
[20] Khaji N, Shafiei M, Jalalpour M. Closed-form solutions for crack detection problem of Timoshenko beams with various boundary conditions. Int J Mech Sci 2009;51:667–81.
[21] Lin HP. Direct and inverse methods on free vibration analysis of simply supported beams with a crack. Eng Struct 2004;26:427–36.
[22] Zhang Z, Huang X, Zhang Z, Hua H. On the transverse vibration of Timoshenko double-beam systems coupled with various discontinuities. Int J Mech Sci 2014;89:222–41.
[23] IEEE, 754-2008-IEEE Standard for Floating-Point Arithmetic;2008.
[24] Low KH. A reliable algorithm for solving frequency equations involving transcendental functions. J Sound Vib 1993;161:369–77.
[25] Horr AM, Schmidt LC. Closed-form solution for the Timoshenko beam theory using a computer-based mathematical package. Comput Struct 1995;55:405–12.
[26] Yu CP, Roesset JM. Dynamic stiffness matrices for linear members with distributed mass. Tamkang J Sci Eng 2001;4:253–64.
[27] Erdelyi NH, Hashemi SM. A dynamic stiffness element for free vibration analysis of delaminated layered beams. Modell Simulat Eng 2012;2012:492415.