A NEW GENERAL FRACTIONAL-ORDER DERIVATIVE WITH RABOTNOV FRACTIONAL-EXPONENTIAL KERNEL

by

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In this article, a general fractional-order derivative of the Riemann-Liouville type with the non-singular kernel involving the Rabotnov fractional-exponential function is addressed for the first time. A new general fractional-order derivative model for the anomalous diffusion is discussed in detail. The general fractional-order derivative operator formula is as a novel and mathematical approch proposed to give the generalized presentation of the physical models in complex phenomena with power law.

Key words: *anomalous diffusion, general fractional-order derivataive, Rabotnov fractional-exponential function, non-singular kernel, power law*

Introduction

General fractional calculus (GFC) [1-4], as a general version of FC acting on the singular (power-law) kernel containing the pioneers, *e.g.*, Liouville [5], Riemann [6], Weyl [7], Sonine [8], Caputo [9] and others (see [1]), has been successfully applied to describe some physical processes in complex phenomena. The general fractional-order derivatives (FDs) and general fractional-order integrals (FIs) with the non-singular kernels of the functions, such as the exponential function[10], Miller-Ross function[11], Lorenzo-Hartley function[12], Gorenflo-Mainardi function[13], Bessel function[14], Mittag-Leffler function[15], Wiman function[16], Prabhakar function[17], sinc function[18], and others [19].

In 1948, the fractional exponential function (also called the Rabotnov fractional exponential (RFE) function[1]) was proposed by Rabotnov [20] and developed to model the internal friction given in [21]. The general FDs in the sense of the Liouville-Caputo type with the non-singular kernel of the RFE function was reported in [22]. However, the general FDs in sense of the Riemann-Liouville type with the non-singular kernel of the RFE function have not been considered to the best of our knowledge.

By the motivation of the tasks involving the physical phenomena with power-law and

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complex behaviors following the RFE function, the target of the paper is to derive the general FDs of the Riemann-Liouville type with the non-singular kernel involving the RFE function and their properties, and to present a general FD model for the anomalous diffusion. The structure of the article is designed as follows. In Section 2, the concepts and properties of the general FDs involving the RFE function are introduced. In Section 3, a typical example for the general FD diffusion model is considered. Finally, the conclusion is drawn in Section 4.

A new GFC of Riemann-Liouville type with the RFE kernel

Suppose that $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}_0^+$, $\mathbb{N}$ and $\mathbb{N}_0$ are the sets of complex numbers, real numbers, non-negative real numbers, positive integers and $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ respectively. Let $L(a,b)$ be the set of those Lebesgue measurable functions on a finite interval (a,b) $(-\infty \le a \le b \le +\infty)$ (for more details, see[1,14]). Suppose that $AC(a,b)$ $(-\infty \le a \le b \le +\infty)$ and $AC^{\kappa}(a,b)$ $(-\infty \le a \le b \le +\infty)$ are the Kolmogorov-Fomin condition (see[1,23]) and the Samko-Kilbas-Marichev condition (see [1,14]), respectively.

General FIs with the RFE kernel

The general FI with the RFE kernel in the interval on a finite interval (a,b) $(-\infty \le a \le b \le +\infty)$ is given as:

$$\left({}_a\mathbb{I}_\tau^{(\alpha)}\Pi\right)(\tau)={}_a\mathbb{I}_\tau^{(\alpha)}\Pi(\tau)=\int_a^\tau \mathrm{M}_\alpha\left(-\gamma(\tau-t)^\alpha\right)\Pi(t)\,dt, \tag{1}$$

where $\Pi \in L(a,b)$, $\gamma \in \mathbb{R}_0^+$, and the RFE function is defined in [1,20-22] as

$$\mathrm{M}_\alpha\left(-\gamma t^\alpha\right)=\sum_{\rho=0}^{\infty}\frac{(-\gamma)^\rho t^{(\rho+1)(\alpha+1)-1}}{\Gamma\left((\rho+1)(\alpha+1)\right)} \tag{2}$$

with $\rho \in \mathbb{N}_0$.

From (1) we have[22]

$$\left({}_0\mathbb{I}_\tau^{(\alpha)}\Pi\right)(\tau)={}_0\mathbb{I}_\tau^{(\alpha)}\Pi(\tau)=\int_0^\tau M_\alpha\left(-\gamma(\tau-t)^\alpha\right)\Pi(t)\,dt, \tag{3}$$

where $a=0$, $\Pi \in L(a,b)$ and $\gamma \in \mathbb{R}_0^+$, and from [22], we have

$$\left(\mathbb{I}_+^{(\alpha)}\Pi\right)(\tau)=\mathbb{I}_+^{(\alpha)}\Pi(\tau)=\int_{-\infty}^\tau M_\alpha\left(-\gamma(\tau-t)^\alpha\right)\Pi(t)\,dt, \tag{4}$$

where $\Pi \in L(-\infty,b)$ and $\gamma \in \mathbb{R}_0^+$, and from [22] we have

$$\left({}_0\mathbb{I}_{+\infty}^{(\alpha)}\Pi\right)(\tau)={}_0\mathbb{I}_{+\infty}^{(\alpha)}\Pi(\tau)=\int_0^{+\infty} M_\alpha\left(-\gamma(\tau-t)^\alpha\right)\Pi(t)\,dt. \tag{5}$$

where $\Pi \in L(0,+\infty)$ and $\gamma \in \mathbb{R}_0^+$.

General FDs of the Liouville-Caputo type with the RFE kernel

The left-sided general FD of the Liouville-Caputo type without the singular kernel of the RFE function on a finite interval (a,b) is given in [22] as:

$$\left({}_{a}^{LC}\mathbb{D}_{\tau}^{(\alpha)}\Pi\right)(\tau)={}_{a}^{LC}\mathbb{D}_{\tau}^{(\alpha)}\Pi(\tau)={}_{a}\mathbb{I}_{\tau}^{(\alpha)}\left(\Pi^{(1)}(\tau)\right)=\int_{a}^{\tau}M_{\alpha}\left(-\gamma(\tau-t)^{\alpha}\right)\Pi^{(1)}(t)\,dt\,, \tag{6}$$

andthe right-sided general FD of the Liouville-Caputo type with the non-singular kernel of the RFE function on a finite interval (a,b) as [22]

$$\left({}_{\tau}^{LC}\mathbb{D}_{b}^{(\alpha)}\Pi\right)(\tau)={}_{\tau}^{LC}\mathbb{D}_{b}^{(\alpha)}\Pi(\tau)={}_{\tau}\mathbb{I}_{b}^{(\alpha)}\left(\Pi^{(1)}(\tau)\right)=-\int_{\tau}^{b}M_{\alpha}\left(-\gamma(t-\tau)^{\alpha}\right)\Pi^{(1)}(t)\,dt\,, \tag{7}$$

where $\Pi \in AC(a,b)$ and $\gamma \in \mathbb{R}_0^+$.

For $\alpha = 1$ we have the same results as in [1,16].

The left-sided general FD of the Liouville-Caputo type without the singular kernel of the RFE function on a finite interval (a,b) is given as in [22] as:

$$\left({}_{a}^{LC}\mathbb{D}_{\tau}^{(\alpha,n)}\Pi\right)(\tau)={}_{a}^{LC}\mathbb{D}_{\tau}^{(\alpha,n)}\Pi(\tau)={}_{a}\mathbb{I}_{\tau}^{(\alpha)}\left(\Pi^{(n)}(\tau)\right)=\int_{a}^{\tau}M_{\alpha}\left(-\gamma(\tau-t)^{\alpha}\right)\Pi^{(n)}(t)\,dt\,, \tag{8}$$

and right-sided general FD of the Liouville-Caputo type with the non-singular kernel of the RFE function on a finite interval (a,b) is given as in [22] as:

$$\left({}_{\tau}^{LC}\mathbb{D}_{b}^{(\alpha,n)}\Pi\right)(\tau)={}_{\tau}^{LC}\mathbb{D}_{b}^{(\alpha,n)}\Pi(\tau)={}_{\tau}\mathbb{I}_{b}^{(\alpha)}\left(\Pi^{(n)}(\tau)\right)=(-1)^{n}\int_{\tau}^{b}M_{\alpha}\left(-\gamma(t-\tau)^{\alpha}\right)\Pi^{(n)}(t)\,dt\,, \tag{9}$$

where $\Pi \in AC^{n}(a,b)$, $n \in \mathbb{N}$ and $\gamma \in \mathbb{R}_0^+$.

For $a = 0$ we have from (6) and (9) that

$$\left({}_{0}^{LC}\mathbb{D}_{\tau}^{(\alpha)}\Pi\right)(\tau)={}_{0}^{LC}\mathbb{D}_{\tau}^{(\alpha)}\Pi(\tau)={}_{0}\mathbb{I}_{\tau}^{(\alpha)}\left(\Pi^{(1)}(\tau)\right)=\int_{0}^{\tau}M_{\alpha}\left(-\gamma(\tau-t)^{\alpha}\right)\Pi^{(1)}(t)\,dt\,, \tag{10}$$

and

$$\left({}_{0}^{LC}\mathbb{D}_{\tau}^{(\alpha,n)}\Pi\right)(\tau)={}_{0}^{LC}\mathbb{D}_{\tau}^{(\alpha,n)}\Pi(\tau)={}_{0}\mathbb{I}_{\tau}^{(\alpha)}\left(\Pi^{(n)}(\tau)\right)=\int_{0}^{\tau}M_{\alpha}\left(-\gamma(\tau-t)^{\alpha}\right)\Pi^{(n)}(t)\,dt\,. \tag{11}$$

The left-sided general FD of the Liouville-Caputo type without the singular kernel of the RFE function on the real axis $\mathbb{R}$ is given as [22]:

$$\left({}^{LC}\mathbb{D}_{+}^{(\alpha)}\Pi\right)(\tau)={}^{LC}\mathbb{D}_{+}^{(\alpha)}\Pi(\tau)=\mathbb{I}_{+}^{(\alpha)}\left(\Pi^{(1)}(\tau)\right)=\int_{-\infty}^{\tau}M_{\alpha}\left(-\gamma(\tau-t)^{\alpha}\right)\Pi^{(1)}(t)\,dt\,, \tag{12}$$

and the right-sided general FD of the Liouville-Caputo type with the non-singular kernel of the RFE function on the real axis $\mathbb{R}$ as [22]

$$\left({}^{LC}\mathbb{D}_{-}^{(\alpha)}\Pi\right)(\tau)={}^{LC}\mathbb{D}_{-}^{(\alpha)}\Pi(\tau)=\mathbb{I}_{-}^{(\alpha)}\left(\Pi^{(1)}(\tau)\right)=-\int_{\tau}^{+\infty}M_{\alpha}\left(-\gamma(t-\tau)^{\alpha}\right)\Pi^{(1)}(t)\,dt, \tag{13}$$

where $\Pi\in AC(-\infty,+\infty)$ and $\gamma\in\mathbb{R}_{0}^{+}$.

The left-sided general FD of the Liouville-Caputo type without the singular kernel of the RFE function on the real axis $\mathbb{R}$ is given as in [22] as:

$$\left({}^{LC}\mathbb{D}_{+}^{(\alpha,n)}\Pi\right)(\tau)={}^{LC}\mathbb{D}_{+}^{(\alpha,n)}\Pi(\tau)=\mathbb{I}_{+}^{(\alpha)}\left(\Pi^{(n)}(\tau)\right)=\int_{-\infty}^{\tau}M_{\alpha}\left(-\gamma(\tau-t)^{\alpha}\right)\Pi^{(n)}(t)\,dt, \tag{14}$$

and right-sided general FD of the Liouville-Caputo type with the non-singular kernel of the RFE function on the real axis $\mathbb{R}$ is given as in [22] as:

$$\left({}^{LC}\mathbb{D}_{-}^{(\alpha)}\Pi\right)(\tau)={}^{LC}\mathbb{D}_{-}^{(\alpha)}\Pi(\tau)=\mathbb{I}_{-}^{(\alpha)}\left(\Pi^{(n)}(\tau)\right)=(-1)^{n}\int_{\tau}^{+\infty}M_{\alpha}\left(-\gamma(t-\tau)^{\alpha}\right)\Pi^{(n)}(t)\,dt, \tag{15}$$

where $\Pi\in AC^{n}(-\infty,+\infty)$, $n\in\mathbb{N}$ and $\gamma\in\mathbb{R}_{0}^{+}$

General FDs of the Riemann-Liouville type with the RFE kernel

The left-sided general FD of the Riemann-Liouville type without the singular kernel of the RFE function on a finite interval (a,b) is defined as

$$\left({}_{a}^{RL}\mathbb{D}_{\tau}^{(\alpha)}\Pi\right)(\tau)={}_{a}^{RL}\mathbb{D}_{\tau}^{(\alpha)}\Pi(\tau)=\frac{d}{d\tau}\left({}_{a}\mathbb{I}_{\tau}^{(\alpha)}\Pi(\tau)\right)=\frac{d}{d\tau}\int_{a}^{\tau}M_{\alpha}\left(-\gamma(\tau-t)^{\alpha}\right)\Pi(t)\,dt, \tag{16}$$

and right-sided general FD of the Riemann-Liouville type with the non-singular kernel of the RFE function on a finite interval (a,b) as

$$\left({}_{\tau}^{RL}\mathbb{D}_{b}^{(\alpha)}\Pi\right)(\tau)={}_{\tau}^{RL}\mathbb{D}_{b}^{(\alpha)}\Pi(\tau)=\frac{d}{d\tau}\left({}_{\tau}\mathbb{I}_{b}^{(\alpha)}\Pi(\tau)\right)=-\frac{d}{d\tau}\int_{\tau}^{b}M_{\alpha}\left(-\gamma(t-\tau)^{\alpha}\right)\Pi(t)\,dt, \tag{17}$$

where $\Pi\in L(a,b)$ and $\gamma\in\mathbb{R}_{0}^{+}$.

The left-sided general FD of the Riemann-Liouville type without the singular kernel of the RFE function on a finite interval (a,b) is defined as

$$\left({}_{a}^{RL}\mathbb{D}_{\tau}^{(\alpha,n)}\Pi\right)(\tau)={}_{a}^{RL}\mathbb{D}_{\tau}^{(\alpha,n)}\Pi(\tau)=\frac{d^{n}}{d\tau^{n}}\left({}_{a}\mathbb{I}_{\tau}^{(\alpha)}\Pi(\tau)\right)=\frac{d^{n}}{d\tau^{n}}\int_{a}^{\tau}M_{\alpha}\left(-\gamma(\tau-t)^{\alpha}\right)\Pi(t)\,dt, \tag{18}$$

andthe right-sided general FD of the Riemann-Liouville type with the non-singular kernel of the RFE function on a finite interval (a,b) as

$$\left({}_{\tau}^{RL}\mathbb{D}_{b}^{(\alpha,n)}\Pi\right)(\tau)={}_{\tau}^{RL}\mathbb{D}_{b}^{(\alpha,n)}\Pi(\tau)=\frac{d^{n}}{d\tau^{n}}\left({}_{\tau}\mathbb{I}_{b}^{(\alpha)}\Pi(\tau)\right)=(-1)^{n}\frac{d^{n}}{d\tau^{n}}\int_{\tau}^{b}M_{\alpha}\left(-\gamma(t-\tau)^{\alpha}\right)\Pi(t)\,dt, \tag{19}$$

where $\Pi \in L(a,b)$, $n \in \mathbb{N}$ and $\gamma \in \mathbb{R}_0^+$.

For $a=0$ we have from (16) and (19) that

$$\left({}_{0}^{RL}\mathbb{D}_{\tau}^{(\alpha)}\Pi\right)(\tau)={}_{0}^{RL}\mathbb{D}_{\tau}^{(\alpha)}\Pi(\tau)=\frac{d}{d\tau}\left({}_{0}\mathbb{I}_{\tau}^{(\alpha)}\Pi(\tau)\right)=\frac{d}{d\tau}\int_{0}^{\tau}M_{\alpha}\left(-\gamma(\tau-t)^{\alpha}\right)\Pi(t)dt, \tag{20}$$

and

$$\left({}_{0}^{RL}\mathbb{D}_{\tau}^{(\alpha,n)}\Pi\right)(\tau)={}_{0}^{RL}\mathbb{D}_{\tau}^{(\alpha,n)}\Pi(\tau)=\frac{d^n}{d\tau^n}\left({}_{0}\mathbb{I}_{\tau}^{(\alpha)}\Pi(\tau)\right)=\frac{d^n}{d\tau^n}\int_{0}^{\tau}M_{\alpha}\left(-\gamma(\tau-t)^{\alpha}\right)\Pi(t)dt. \tag{21}$$

The left-sided general FD of the Riemann-Liouville type without the singular kernel of the RFE function on the real axis $\mathbb{R}$ is defined as

$$\left({}^{RL}\mathbb{D}_{+}^{(\alpha)}\Pi\right)(\tau)={}^{RL}\mathbb{D}_{+}^{(\alpha)}\Pi(\tau)=\frac{d}{d\tau}\left(\mathbb{I}_{+}^{(\alpha)}\Pi(\tau)\right)=\frac{d}{d\tau}\int_{-\infty}^{\tau}M_{\alpha}\left(-\gamma(\tau-t)^{\alpha}\right)\Pi(t)dt, \tag{22}$$

and right-sided general FD of the Riemann-Liouville type with the non-singular kernel of the RFE function on the real axis $\mathbb{R}$ as

$$\left({}^{RL}\mathbb{D}_{-}^{(\alpha)}\Pi\right)(\tau)={}^{RL}\mathbb{D}_{-}^{(\alpha)}\Pi(\tau)=\frac{d}{d\tau}\left(\mathbb{I}_{-}^{(\alpha)}\Pi(\tau)\right)=-\frac{d}{d\tau}\int_{\tau}^{+\infty}M_{\alpha}\left(-\gamma(t-\tau)^{\alpha}\right)\Pi(t)dt, \tag{23}$$

where $\Pi \in L(-\infty,+\infty)$ and $\gamma \in \mathbb{R}_0^+$.

The left-sided general FD of the Riemann-Liouville type without the singular kernel of the RFE function on the real axis $\mathbb{R}$ is defined as

$$\left({}^{RL}\mathbb{D}_{+}^{(\alpha,n)}\Pi\right)(\tau)={}^{RL}\mathbb{D}_{+}^{(\alpha,n)}\Pi(\tau)=\frac{d^n}{d\tau^n}\left(\mathbb{I}_{+}^{(\alpha)}\Pi(\tau)\right)=\frac{d^n}{d\tau^n}\int_{-\infty}^{\tau}M_{\alpha}\left(-\gamma(\tau-t)^{\alpha}\right)\Pi(t)dt, \tag{24}$$

and right-sided general FD of the Riemann-Liouville type with the non-singular kernel of the RFE function on the real axis $\mathbb{R}$ as

$$\left({}^{RL}\mathbb{D}_{-}^{(\alpha)}\Pi\right)(\tau)={}^{RL}\mathbb{D}_{-}^{(\alpha)}\Pi(\tau)=\frac{d^n}{d\tau^n}\left(\mathbb{I}_{-}^{(\alpha)}\Pi(\tau)\right)=(-1)^n\frac{d^n}{d\tau^n}\int_{\tau}^{+\infty}M_{\alpha}\left(-\gamma(t-\tau)^{\alpha}\right)\Pi(t)dt, \tag{25}$$

where $\Pi \in L(-\infty,+\infty)$, $n \in \mathbb{N}$ and $\gamma \in \mathbb{R}_0^+$.

For $\Pi(\tau)\big|_{\tau=0}=\Pi(0)$ there exists

$${}_{a}^{LC}\mathbb{D}_{\tau}^{(\alpha)}\Pi(\tau)={}_{a}^{RL}\mathbb{D}_{\tau}^{(\alpha)}\Pi(\tau)-M_{\alpha}\left(-\gamma\tau^{\alpha}\right)\Pi(0). \tag{26}$$

General FIs via the Prabhakar function

The left-sided general FI of $\Pi(\tau)$ is given as [22]:

$${}_{a}\mathbb{I}_{\tau}^{(\alpha,n)}\Pi(\tau)=\int_{a}^{\tau}\Xi_{\alpha}\left(-\gamma(\tau-t)^{\alpha}\right)\Pi(t)dt=\int_{a}^{\tau}(\tau-t)^{n-(\alpha+2)}E_{\alpha+1,n-(\alpha+1)}^{-1}\left(-\gamma(\tau-t)^{\alpha+1}\right)\Pi(t)dt \tag{27}$$

and the right-sided general FI of $\Pi(\tau)$ as

$$ {}_{\tau}\mathbb{I}_{b}^{(\alpha,n)}\Pi(\tau)=-\int_{\tau}^{b}\Xi_{\alpha}\left(-\gamma(t-\tau)^{\alpha}\right)\Pi(t)dt=-\int_{\tau}^{b}(t-\tau)^{n-(\alpha+2)}E_{\alpha+1,n-(\alpha+1)}^{-1}\left(-\gamma(t-\tau)^{\alpha+1}\right)\Pi(t)dt \quad (28) $$

where $\Pi\in L(a,b)$, $n\in\mathbb{N}$, $\gamma\in\mathbb{R}_{0}^{+}$, and $\Xi_{\alpha}\left(-\gamma\tau^{\alpha}\right)=\tau^{n-(\alpha+2)}H_{\alpha+1,n-(\alpha+1)}^{-1}\left(-\gamma\tau^{\alpha+1}\right)$ with the Prabhakar function, given as in [1,24]:

$$ H_{\alpha,\beta}^{\gamma}(\tau)=\sum_{\rho=0}^{\infty}\frac{\Gamma(\gamma+\rho)}{\Gamma(\rho\alpha+\beta)\Gamma(\gamma)}\frac{\tau^{\rho}}{\Gamma(\rho+1)}. $$

The left-sided general FI of $\Pi(\tau)$ is given as:

$$ \mathbb{I}_{+}^{(\alpha,n)}\Pi(\tau)=\int_{-\infty}^{\tau}\Xi_{\alpha}\left(-\gamma(\tau-t)^{\alpha}\right)\Pi(t)dt=\int_{-\infty}^{\tau}(\tau-t)^{n-(\alpha+2)}E_{\alpha+1,n-(\alpha+1)}^{-1}\left(-\gamma(\tau-t)^{\alpha+1}\right)\Pi(t)dt \quad (29) $$

and the right-sided general FI of $\Pi(\tau)$ as

$$ \mathbb{I}_{-}^{(\alpha,n)}\Pi(\tau)=-\int_{\tau}^{+\infty}\Xi_{\alpha}\left(-\gamma(t-\tau)^{\alpha}\right)\Pi(t)dt=\int_{\tau}^{-\infty}(t-\tau)^{n-(\alpha+2)}E_{\alpha+1,n-(\alpha+1)}^{-1}\left(-\gamma(t-\tau)^{\alpha+1}\right)\Pi(t)dt \quad (30) $$

where $\Pi\in L(-\infty,+\infty)$, $n\in\mathbb{N}$, and $\gamma\in\mathbb{R}_{0}^{+}$.

The properties for the general FDs and FIs are as follows:

(I) Let $\Pi\in L(a,b)$ and $n\in\mathbb{N}$. Then ${}_{0}^{RL}\mathbb{D}_{\tau}^{(\alpha,n)}\left({}_{0}\mathbb{I}_{\tau}^{(\alpha,n)}\Pi(\tau)\right)=\Pi(\tau)$;

(II)Let $\Pi\in(-\infty,+\infty)$ and $n\in\mathbb{N}$. Then ${}^{RL}\mathbb{D}_{+}^{(\alpha,n)}\left(\mathbb{I}_{+}^{(\alpha,n)}\Pi(\tau)\right)=\Pi(\tau)$;

(III) Let $\Pi\in AC^{n}(a,b)$ and $n\in\mathbb{N}$. Then ${}_{0}^{LC}\mathbb{D}_{\tau}^{(\alpha,n)}\left({}_{0}\mathbb{I}_{\tau}^{(\alpha,n)}\Pi(\tau)\right)=\Pi(\tau)$;

(IIII)Let $\Pi\in(-\infty,+\infty)$ and $n\in\mathbb{N}$. Then ${}^{LC}\mathbb{D}_{+}^{(\alpha,n)}\left(\mathbb{I}_{+}^{(\alpha,n)}\Pi(\tau)\right)=\Pi(\tau)$.

The Laplace transforms of the general FDs are as follows:

$$ G\left[{}_{0}^{RL}\mathbb{D}_{\tau}^{(\alpha)}\Pi(\tau)\right]=s^{-\alpha}\left(1+\lambda s^{-(\alpha+1)}\right)^{-1}\Pi(s)-{}_{0}\mathbb{I}_{\tau}^{(\alpha,1)}\Pi(0) \quad (31) $$

and

$$ G\left[{}_{0}^{RL}\mathbb{D}_{\tau}^{(\alpha,n)}\Pi(\tau)\right]=s^{n-(\alpha+1)}\left(1+\lambda s^{-(\alpha+1)}\right)^{-1}\Pi(s)-\sum_{\eta=0}^{n-1}s^{n-\eta-1}\left(\frac{d^{\eta}}{d\tau^{\eta}}\left[{}_{0}\mathbb{I}_{\tau}^{(\alpha,n)}\Pi(0)\right]\right), \quad (32) $$

where the Laplace transform of $g(\tau)$ is [1]

$$ G[g(\tau)]=g(s)=\int_{0}^{\infty}e^{-s\tau}g(\tau)d\tau \quad (33) $$

with $s\in\mathbb{C}$.

For ${}_{0}\mathbb{I}_{\tau}^{(\alpha,1)}\Pi(0)=0$ we have from (31) that

$$G\left[{}_{0}^{RL}\mathbb{D}_{\tau}^{(\alpha)}\Pi(\tau) \right] = s^{-\alpha}\left(1+\lambda s^{-(\alpha+1)}\right)^{-1}\Pi(s). \quad (34)$$

A general FD diffusion model with the RFE kernel

We now consider the anomalous diffusion model containing the general FDs of the Riemann-Liouville type with the RFE kernel

$$ {}_{0}^{RL}\partial_{\tau}^{(\alpha)}\psi(x,\tau) = \xi \frac{\partial^2 \psi(x,\tau)}{\partial x^2} \quad (36)$$

with the initial condition ${}_{0}\mathbb{I}_{\tau}^{(\alpha,1)}\psi(x,0)=0$ and the boundary conditions: $\psi(0,\tau)=1$, $\psi(x,\tau)\to 0, x\to\infty, \tau>0$, where ξ is the diffusivity constant, and

$$ {}_{0}^{RL}\partial_{\tau}^{(\alpha)}\psi(x,\tau) = \frac{\partial}{\partial\tau}\int_{0}^{\tau} M_{\alpha}\left(-\gamma(\tau-t)^{\alpha}\right)\psi(x,\tau)\,dt. \quad (37)$$

With the use of the Laplace transform of (36) with respect to the variable τ, we can get

$$\frac{d^2\psi(x,s)}{dx^2} = \frac{s^{-\alpha}\left(1+\lambda s^{-(\alpha+1)}\right)^{-1}}{\xi}\psi(x,s), \quad (38)$$

which, due to the boundary conditions, this implies that

$$\psi(x,s) = e^{-x\sqrt{\frac{s^{-\alpha}\left(1+\lambda s^{-(\alpha+1)}\right)^{-1}}{\xi}}} = \sum_{\upsilon=0}^{\infty}\frac{\left(-\frac{x}{\sqrt{\xi}}\right)^{\upsilon}}{\Gamma(1+\upsilon)} s^{-\upsilon\alpha}\left(1+\lambda s^{-(\alpha+1)}\right)^{-\upsilon}. \quad (39)$$

The general solution for (36) can be represented as follows:

$$\psi(x,t) = \sum_{\upsilon=0}^{\infty}\frac{\left(-\frac{x}{\sqrt{\xi}}\right)^{\upsilon}}{\Gamma(1+\upsilon)}\left(\sum_{\rho=0}^{\infty}\frac{\Gamma(\upsilon+\rho)}{\Gamma(\rho\alpha+\upsilon\alpha)\Gamma(\upsilon)}\frac{(-\lambda)^{\rho}\tau^{(\alpha+1)\rho}}{\Gamma(\rho+1)}\right) = \sum_{\upsilon=0}^{\infty}\frac{\left(-\frac{x}{\sqrt{\xi}}\right)^{\upsilon}}{\Gamma(1+\upsilon)}\tau^{\upsilon\alpha-1}H_{\alpha+1,\upsilon\alpha}^{\upsilon}\left(-\lambda\tau^{\alpha+1}\right). \quad (40)$$

Conclusion

In the present work, we proposed the general FD of the Riemann-Liouville type with the non-singular kernel involving the RFE function. With the aid of the presented Laplace transforms, the general FD model for the anomalous diffusion with the solutions containing the Prabhakar function was investigated in detail. The formula of the general FD of the Riemann-Liouville type can be given to explore the mathematical models in physics and engineering practice.

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Nomenclature

α - fractional order, [-]	x - space coordinate, [m]
ξ - diffusivity constant, [m^2/s]	τ - time, [s]

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