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Original research article

Comments on "Soliton solutions to fractional-order nonlinear differential equations based on the exp-function method"

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ABSTRACT

In their recently published article, Guner and Atik apply the exp-function method to derive solitary solutions to fractional-order nonlinear differential equations. We argue that the solutions provided by Guner and Atik do not satisfy the considered equations. Furthermore, we derive correct solitary solutions as well as necessary and sufficient conditions for the existence of these solutions in the space of equation parameters and initial conditions.

1. Introduction

In their recent article [1] Guner and Atik use the exp-function method to derive exact solitary solutions to the time fractional modified nonlinear Kawahara equation and the nonlinear fractional advection–diffusion–reaction equation. One of the objectives of this article is to demonstrate that the solutions obtained in [1] do not satisfy the considered differential equations. Furthermore, we derive the correct solitary solutions as well as conditions of existence of these solutions in the space of equation parameters and initial conditions.

2. Preliminaries

2.1. Guner and Atik's results

In their paper [1], Guner and Atik consider the following fractional-order nonlinear differential equations:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + u^2\frac{\partial u}{\partial x} + \varepsilon\frac{\partial^2 u}{\partial x^2} + \delta\frac{\partial^3 u}{\partial x^3} = 0, \tag{1}$$

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which is the time fractional modified nonlinear Kawahara equation; and

$$\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial u}{\partial x} - \frac{1}{10}\frac{\partial^2 u}{\partial x^2} - \frac{1}{2}u\left(u - \frac{1}{2}\right)(u-1) = 0 \tag{2}$$

that describes nonlinear fractional advection-diffusion-reaction processes. The notation $\frac{\partial^\alpha u}{\partial t^\alpha}$ represents the Riemann–Liouville fractional derivative:

$$\frac{\partial^\alpha}{\partial t^\alpha}f(t) = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{f(\xi)-f(0)}{\left(t-\xi\right)^\alpha}\,\mathrm{d}\xi, \tag{3}$$

where $0<\alpha<1$. The authors of [1] use the variable substitution

$$u(x,t) = U(\xi), \quad \xi = kx - \frac{ct^\alpha}{\Gamma(1+\alpha)}, \quad c \neq 0, k \neq 0; \tag{4}$$

to transform (1) into the ordinary differential equation (ODE):

$$-clU + k\frac{U^3}{3} + \varepsilon k^2 U' + \delta k^3 U'' + \zeta_0 = 0, \tag{5}$$

where $l, \zeta_0 \epsilon \mathbb{R}$.

Analogously, (2) is transformed into the ODE:

$$(lc+k)U' + \frac{k^2}{10}U'' + \frac{1}{2}U\left(U - \frac{1}{2}\right)(U-1) = 0, \tag{6}$$

where $l \epsilon \mathbb{R}$.

2.2. Rearrangement of (5) and (6)

Denoting

$$b = \frac{\varepsilon}{\delta k}; \quad c_0 = -\frac{\zeta_0}{\delta k^3}; \quad c_1 = \frac{cl}{\delta k^3}; \quad c_2 = 0; \quad c_3 = -\frac{1}{3\delta k^2}, \tag{7}$$

transforms (5) into:

$$U'' + bU' = c_0 + c_1 U + c_2 U^2 + c_3 U^3, \tag{8}$$

where $b, c_k \epsilon \mathbb{R}, \quad k = 0, \ldots, 3$.

Analogously, denoting

$$b = \frac{10(lc+k)}{k^2}; \quad c_0 = 0; \quad c_1 = -\frac{5}{2k^2}; \quad c_2 = \frac{15}{2k^2}; \quad c_3 = -\frac{5}{k^2}, \tag{9}$$

transforms (6) into (8).

The solution $U(\xi)$ satisfies initial conditions:

$$U(\xi_0) = U_0, \quad U'_\xi|_{\xi=\xi_0} = U'_0, \tag{10}$$

where $\xi, U_0, U'_0 \epsilon \mathbb{R}$.

Note that (8) is a special case of the Hodgkin–Huxley equation [2,3].

3. Validity of Guner and Atik's solutions

In [1], Guner and Atik seek to construct solitary solutions to (5) and (6) that have the following analytic expression:

$$U(\xi) = \frac{a_1\exp(\xi) + a_0 + a_{-1}\exp(-\xi)}{b_1\exp(\xi) + b_0 + b_{-1}\exp(-\xi)}, \tag{11}$$

where $a_k, b_k \epsilon \mathbb{C}, \quad k = -1, 0, 1$. It can be noted that some of the solutions provided in [1] do not satisfy Eqs. (5) and (6). For example, the following solution to (5) is given in Section 3, case (c):

$$U = \frac{\pm\frac{(3k\delta-\varepsilon)b_1}{\sqrt{-6\delta}}\exp(\xi) \pm \frac{(3k\delta+\varepsilon)b_0}{\sqrt{-6\delta}}}{b_1\exp(\xi) + b_0}, \tag{12}$$

where $b_0, b_1, k, \varepsilon, l, \delta \epsilon \mathbb{C}$ and

$$c = -\frac{(3k^2\delta^2 + \varepsilon^2)k}{6\delta l}, \quad \zeta_0 = \mp\frac{\varepsilon\delta(9k^2\delta^2 - \varepsilon^2)}{9\delta\sqrt{-6\delta}}. \tag{13}$$

However, inserting (12) and (13) into (5) yields:

$$\frac{6\varepsilon}{\left(-6\delta\right)^{\frac{3}{2}}(b_1\exp(\xi)+b_0)}\left(k^2\delta^2-\frac{\varepsilon^2}{9}\right)\left(-b_1(k-\delta)\exp(3\xi)-3b_0b_1^2(k-\delta)\exp(2\xi)+3b_0^2b_1(k+\delta)\exp(\xi)+b_0^3(k+\delta)\right)=0. \quad (14)$$

But it is clear that (14) does not hold true.

It can also be noted that the "solutions" provided in Section 4, cases (g), (i) and (j) of [1] do not satisfy (6).

In the following sections, we will derive correct solitary solutions to (5), (6) and demonstrate that these solutions do not hold for all equation parameters and initial conditions.

4. Construction of kink solitary solutions to (5) and (6)

4.1. Riccati equation and its solution

The Riccati ordinary differential equation with constant coefficients reads:

$$y'=\alpha_0+\alpha_1 y+\alpha_2 y^2; \quad y(\xi_0)=y_0, \quad (15)$$

where $\alpha_2 \neq 0$, $\alpha_1, \alpha_0 \epsilon \mathbb{R}$. The Riccati equation (15) admits the kink solution [4,5]:

$$y=\frac{y_2(y_0-y_1)\exp(\alpha_2 y_1(\xi-\xi_0))-y_1(y_0-y_2)\exp(\alpha_2 y_2(\xi-\xi_0))}{(y_0-y_1)\exp(\alpha_2 y_1(\xi-\xi_0))-(y_0-y_2)\exp(\alpha_2 y_2(\xi-\xi_0))}, \quad (16)$$

where y_1, y_2 are roots of the polynomial $\alpha_0+\alpha_1 y+\alpha_2 y^2$.

Note that the solution (16) holds for all initial conditions ξ_0, y_0.

4.2. Extended Riccati equation

Differentiating the Riccati equation (15) yields:

$$y''=\alpha_0\alpha_1+(2\alpha_0\alpha_2+\alpha_1^2)y+3\alpha_1\alpha_2 y^2+2\alpha_2^2 y^3. \quad (17)$$

Multiplying (15) by $\beta\epsilon\mathbb{R}$ and adding to (17) results in:

$$y''+\beta y'=\beta\alpha_0+\alpha_1\alpha_2+(2\alpha_0\alpha_2+\alpha_1^2+\beta\alpha_1)y+(3\alpha_1\alpha_2+\beta\alpha_2)y^2+2\alpha_2^2 y^3. \quad (18)$$

Note that (8) and (18) coincide if the following conditions are satisfied:

$$\beta=b, \quad \beta\alpha_0+\alpha_1\alpha_2=c_0; \quad (19)$$

$$2\alpha_0\alpha_2+\alpha_1^2+\beta\alpha_1=c_1; \quad (20)$$

$$3\alpha_1\alpha_2+\beta\alpha_2=c_2; \quad (21)$$

$$2\alpha_2^2=c_3. \quad (22)$$

4.3. Construction of kink solitary solution to (8)

Corollary 4.1. *Let (19)–(22) hold true. Every solution to (15) is also a solution to (8). However, a solution to (8) does satisfy (15) if and only if the following constraint on initial conditions (10) holds true:*

$$U_0'=\alpha_0+\alpha_1 U_0+\alpha_2 U_0^2. \quad (23)$$

Thus:

$$U(\xi)|_{U_0'=\alpha_0+\alpha_1 U_0+\alpha_2 U_0^2}=y(\xi). \quad (24)$$

A detailed proof of this statement can be found in [2].

Note that constraint (23) on the initial conditions must hold true for both (5) and (6). However, conditions (19)–(22) have different forms with respect to the parameters of (5) and (6), which are given in the following subsections.

4.4. Construction of kink solitary solutions to (5)

Eq. (7) together with (19)–(22) yields that (5) admits the kink solitary solution (16) if parameters α_0, α_1, α_2, β satisfy following equalities:

$$\alpha_0=-\frac{3\zeta_0}{\varepsilon k^2}, \quad \alpha_1=-\frac{\varepsilon}{3\delta k}; \quad (25)$$

$$\alpha_2 = \frac{1}{\pm\sqrt{\delta}k}, \quad \beta = \frac{\varepsilon}{\delta k}, \tag{26}$$

and parameters of (5) satisfy the relationship:

$$\mp 27\delta^{\frac{3}{2}}\zeta_0 - 18cl\delta\varepsilon - 4\varepsilon^3 k = 0. \tag{27}$$

Thus (16) is a solution to (5) if and only if (25)–(27) hold true and initial conditions satisfy (23).

4.5. *Construction of kink solitary solutions to (6)*

Comparing expressions (19)–(22) and (9) yields that (6) admits the kink solitary solution (16) if the equalities hold true:

$$\alpha_0 = 0, \quad \alpha_1 = -\frac{10(lc+k)}{k^2}; \tag{28}$$

$$\alpha_2 = -\frac{3}{8(lc+k)}, \quad \beta = \frac{10(lc+k)}{k^2}, \tag{29}$$

and parameters of (6) satisfy the following constraint:

$$(lc+k)^2\left(l^2c^2 + 2ckl + \frac{169}{160}k^2\right) = 0. \tag{30}$$

The kink solitary solution (16) satisfies (6) if and only if (28)–(30) and (23) hold true.

5. Construction of bright and dark solitary solutions to (5) and (6)

It has been demonstrated in [3,6] that bright and dark solitary solutions to (8) exist only in one of the following cases:

1. $c_3 = 0$;
2. $b = 0$.

Note that the case $c_3 = 0$ is not applicable for Eqs. (5) and (6), because the parameter c_3 can only be equal to zero when k or δ tends to infinity – which invalidates the original equations. Thus, only the case $b = 0$ needs to be considered in this paper. Note that (8) with $b = 0$ can be rewritten as:

$$U'' = c_3(U - U_1)(U - U_2)(U - U_3), \tag{31}$$

where U_1, U_2, U_3 are roots of the polynomial $c_0 + c_1U + c_2U^2 + c_3U^3$.

5.1. *Generating equation of bright and dark solitary solutions*

Let us consider the first order differential equation:

$$z' = \nu(z - z_1)\sqrt{(z - z_2)(z - z_3)}, \quad z(\xi_0) = z_0; \tag{32}$$

where $\nu, z_1 \in \mathbb{R}$; $z_2, z_3 \in \mathbb{C}$.

Eq. (32) admits the following bright and dark solitary solution [6]:

$$z(\xi) = \frac{\sigma_2\exp(2\nu(\xi - \xi_0)) + \sigma_1\exp(\nu(\xi - \xi_0)) + \sigma_0}{(\exp(\nu(\xi - \xi_0)) - \lambda_1)(\exp(\nu(\xi - \xi_0)) - \lambda_2)}, \tag{33}$$

where

$$\nu = \pm\sqrt{(z_1 - z_2)(z_1 - z_3)}; \tag{34}$$

$$\begin{aligned} \lambda_1 &= 1 + \frac{2(z_1 - z_2)(z_1 - z_3)}{(\pm\sqrt{(z_0 - z_2)(z_0 - z_3)} + z_0 - z_1)\sqrt{(z_1 - z_2)(z_1 - z_3)} - (z_1 - z_2)(z_1 - z_3)}; \\ \lambda_2 &= 1 + \frac{2(z_1 - z_2)(z_1 - z_3)}{(\pm\sqrt{(z_0 - z_2)(z_0 - z_3)} - z_0 + z_1)\sqrt{(z_1 - z_2)(z_1 - z_3)} - (z_1 - z_2)(z_1 - z_3)}, \end{aligned} \tag{35}$$

and

$$\sigma_2 = z_1; \quad \sigma_1 = -(z_1(\lambda_1 + \lambda_2) + (\lambda_1 - 1)\mu_2 + (\lambda_2 - 1)\mu_3); \quad \sigma_0 = z_1\lambda_1\lambda_2 + (\lambda_1 - 1)\mu_2\lambda_1 + (\lambda_2 - 1)\mu_3\lambda_2. \tag{36}$$

The coefficients μ_2, μ_3 depend on z_0, z_1, z_2, z_3 and are omitted for brevity; their explicit expressions can be found in [6].

5.2. Extended equation

Differentiation of (32) yields the equation

$$z'' = 2\nu^2(z - z_1)\left(z^2 - \frac{1}{4}(2z_1 + 3z_2 + 3z_3)z + \frac{1}{4}(z_1z_2 + z_1z_3 + 2z_2z_3)\right). \tag{37}$$

It can be observed that Eqs. (37) and (31) coincide if the following conditions hold true:

$$c_3 = 2\nu^2, \quad U_1 = z_1, \quad U_2 + U_3 = \frac{1}{4}(2z_1 + 3z_2 + 3z_3); \tag{38}$$

$$U_2U_3 = \frac{1}{4}(z_1z_2 + z_1z_3 + 2z_2z_3). \tag{39}$$

5.3. Construction of bright and dark solitary solutions to (31)

A similar statement to Corollary 4.1 can be formulated with respect to Eq. (32):

Corollary 5.1. *Let (38) and (39) hold true. All solutions to (32) also satisfy (31), but solutions to (31) satisfy (32) if and only if the initial condition constraint*

$$U_0' = \nu(U_0 - z_1)\sqrt{(U_0 - z_2)(U_0 - z_3)}; \tag{40}$$

holds true. Note that

$$U(\xi)|_{U_0' = \nu(U_0 - z_1)\sqrt{(U_0 - z_2)(U_0 - z_3)}} = z(\xi). \tag{41}$$

The proof of this statement can be found in [6].

Eq. (41) holds true for both (5) and (6) only if the constraint (40) is imposed. However, the relationship between the parameters of (5) and (6) can be specified more concretely than in (38) and (39). These relations are given in the following subsections.

5.4. Construction of bright and dark solitary solutions to (5)

According to (7) the solution to Eqs. (38) and (39) with respect to δ, ε, k, ζ_0 reads:

$$\delta = -\frac{\left(z_1^2 + 2z_1z_2 + 2z_1z_2 + z_2z_3\right)^2}{216\nu^2c^2l^2}, \quad \varepsilon = 0; \tag{42}$$

$$k = -\frac{6cl}{z_1^2 + 2z_1z_2 + 2z_1z_2 + z_2z_3}, \quad \zeta_0 = \frac{clz_1(z_1z_2 + z_1z_3 + 2z_2z_3)}{2(z_1^2 + 2z_1z_2 + 2z_1z_2 + z_2z_3)}, \tag{43}$$

where $c, l \in \mathbb{C}$ and z_1, z_2, z_3 satisfy:

$$2z_1 + z_2 + z_3 = 0. \tag{44}$$

Relation (44) yields $z_3 = -2z_1 - z_2$. Then Eqs. (42) and (43) are transformed into:

$$\delta = -\frac{\left(3z_1^2 + 2z_1z_2 + z_2^2\right)^2}{216\nu^2c^2l^2}, \quad \varepsilon = 0; \tag{45}$$

$$k = \frac{6cl}{3z_1^2 + 2z_1z_2 + z_2^2}, \quad \zeta_0 = \frac{clz_1\left(z_1^2 + 2z_1z_2 + z_2^2\right)}{3z_1^2 + 2z_1z_2 + z_2^2}, \quad c, l \in \mathbb{C}. \tag{46}$$

Thus (5) admits the solitary solution (33) if and only if (45), (46) and (41) hold true.

5.5. Construction of bright and dark solitary solutions to (6)

According to (9), relations (38) and (39) can be solved for k, c:

$$c = \pm\frac{\sqrt{10}i}{2\nu l}, \quad k = \mp\frac{\sqrt{10}i}{2\nu}, \quad l \in \mathbb{C}. \tag{47}$$

However, (47) holds true only if:

$$10 - 10z_1 - 5z_2 - 5z_3 = 0; \tag{48}$$

$$10 + 5(z_2 + z_3)z_1^2 + 10z_1z_2z_3 = 0; \tag{49}$$

$$z_1^2+2(z_2+z_3)z_1+z_2z_3-1=0. \tag{50}$$

Eqs. (48)–(50) yield solutions:

$$z_1=\frac{1}{107}\left(-8\omega^5+92\omega^4-162\omega^3+57\omega^2+166\omega-32\right); \tag{51}$$

$$z_2=\frac{1}{107}\left(16\omega^5-184\omega^4+324\omega^3-114\omega^2-439\omega+278\right), \quad z_3=\omega, \tag{52}$$

where ω satisfies:

$$4\omega^6-12\omega^5+11\omega^4+18\omega^3-31\omega^2+6\omega+29=0. \tag{53}$$

Note that explicit analytic solutions to (53) do not exist. Also, note that c, k are complex parameters in the general case, but the coefficients of (6) are real, since

$$lc+k=0, \quad k^2=\pm\frac{10}{4\nu^2}. \tag{54}$$

Thus (6) admits the solitary solution (33) if and only if (47), (51), (52) and (41) hold true.

6. Computational experiments

6.1. Construction of kink solitary solutions to (5)

Let us consider the following equation:

$$U''+\sqrt{2}U'=\frac{\sqrt{2}}{2}-\frac{43}{46}U+\frac{1}{2}U^3, \quad U(\xi_0)=U_0, \quad U'_\xi|_{\xi=\xi_0}=U'_0. \tag{55}$$

Note that (55) coincides with (5) for $\delta=1$; $\varepsilon=2$; $k=\sqrt{2}$; $\zeta_0=-2$; $l=43$; $c=-\sqrt{218}$. It can be observed that condition (27) is satisfied. Thus, according to (25) and (26):

$$\alpha_0=\frac{3}{4}, \quad \alpha_1=-\frac{\sqrt{2}}{3}, \quad \alpha_2=-\frac{1}{2}, \quad \beta=\sqrt{2}. \tag{56}$$

Eq. (16) yields that (55) admits the following kink solitary solution:

$$y(\xi,\xi_0,U_0)=\frac{\left(\left(2\sqrt{2}+\sqrt{62}\right)U_0-9\right)\exp\left(\frac{2\sqrt{2}-\sqrt{62}}{12}(\xi-\xi_0)\right)+\left(\left(\sqrt{62}-2\sqrt{2}\right)U_0+9\right)\exp\left(\frac{2\sqrt{2}+\sqrt{62}}{12}(\xi-\xi_0)\right)}{\left(\sqrt{62}-2\sqrt{2}-6U_0\right)\exp\left(\frac{2\sqrt{2}-\sqrt{62}}{12}(\xi-\xi_0)\right)+6\left(\sqrt{62}+2\sqrt{2}+6U_0\right)\exp\left(\frac{2\sqrt{2}+\sqrt{62}}{12}(\xi-\xi_0)\right)}, \tag{57}$$

if and only if the initial conditions of (55) satisfy constraint (24):

$$U'_0=\frac{3}{4}-\frac{\sqrt{2}}{3}U_0-\frac{1}{2}U_0^2. \tag{58}$$

The kink solitary solution (57) is illustrated in Fig. 1.

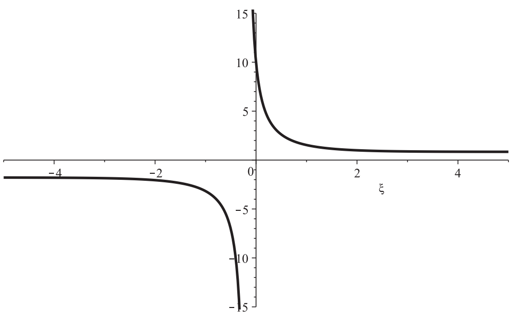


Fig. 1. The kink solitary solution (57) to (55) with $\xi_0=0$, $s=10$.

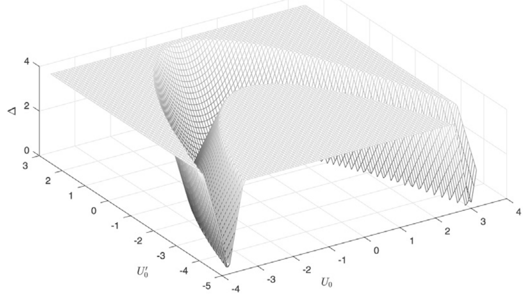


Fig. 2. Plot of the error Δ between the numerical solution to (55) and (57) for $-4 \le U_0 \le 3.2$; $-5 \le U_0' \le 2.5$. The error is estimated with $N = 75$ steps with a step-size $h = 0.001$. Errors higher than 4 are truncated to 4 for clarity. Note that the error is almost zero on the curve defined by (58).

Numerical integration can be used to illustrate constraint (58). Using a constant-step, time-forward integrator, the approximate solution to (55) (denoted as $\widehat{U}\left(\xi, \xi_0, U_0, U_0'\right)$) with initial conditions ξ_0, U_0, U_0' is obtained. Next, the absolute error between the approximate numerical solution $\widehat{U}(\xi, \xi_0, U_0, U_0')$ and the analytical solitary solution (57) is estimated:

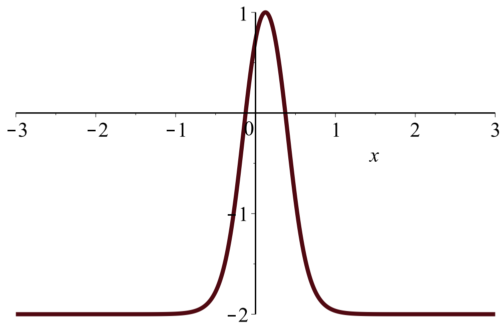
$$\Delta(\xi_0, U_0, U_0') := \sum_{j=1}^{N} \left| \widehat{U}(\xi_0 + jh, \xi_0, U_0, U_0') - y(\xi_0 + jh, \xi_0, U_0) \right|, \tag{59}$$

where N is the number of time-forward integration steps; h is the step-size. If the solution (57) would hold for all initial conditions $\left(\xi_0, U_0, U_0'\right)$, the error $\Delta\left(\xi_0, U_0, U_0'\right)$ would be zero. However, it can be seen from Fig. 2 that this is not true and that the errors are near-zero only on the curve defined by (57).

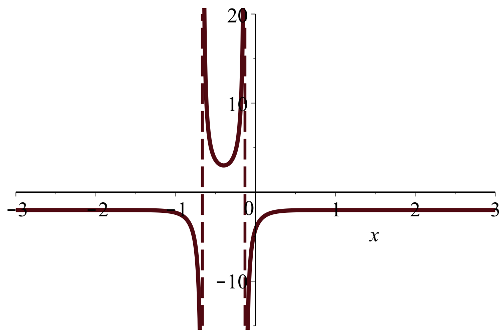
6.2. *Construction of bright and dark solitary solutions to (5)*

Let us consider the following equation:

$$U'' = 8U^3 - 36U + 8; \quad U(\xi_0) = U_0, \quad U'_{\xi}|_{\xi=\xi_0} = U'_0. \tag{60}$$



(a)



(b)

Fig. 3. Solitary solutions to (60) with $\xi_0 = 0$, $U_0 = 4$ (part (a)) and $U_0 = -1$ (part (b)). Dashed lines denote singularity points.

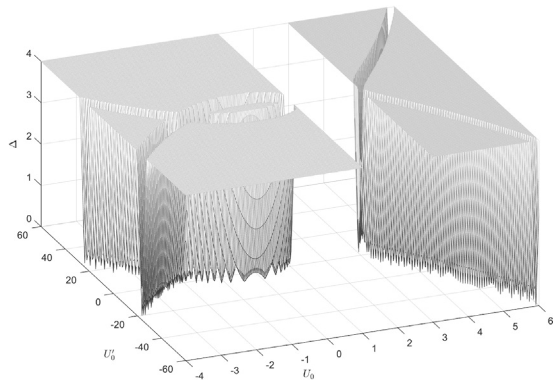


Fig. 4. Plot of the error Δ between the numerical solution to (60) and the analytical solitary solution to (62) for $-4 \le U_0 \le 6$; $-60 \le U_0' \le 60$. The error is estimated with $N = 30$ steps; the step-size h is 0.002. Errors higher than 4 are truncated to 4 for clarity. The region $1 < U_0 < 3$ is unfilled because the constraint (63) cannot be satisfied there. Note that the error is almost zero on the curve defined by (58).

Eq. (60) corresponds to (5) with $\delta = -\frac{3}{32c^2l^2}$, $\varepsilon = 0$, $k = \frac{2}{3}cl$, $\zeta_0 = -\frac{2}{9}cl$; $c, l \in \mathbb{C}$. Note that conditions (42)–(44) are satisfied with the parameters:

$$\nu = \pm 2; \quad z_1 = -2; \quad z_2 = 1; \quad z_3 = 3, \tag{61}$$

thus the solution to (60) corresponds to the solution to the following equation:

$$z' = \pm 2(z+2)\sqrt{(z-1)(z-3)}, \tag{62}$$

if the initial conditions U_0, U_0' satisfy the constraint:

$$U_0' = \pm 2(U_0+2)\sqrt{(U_0-1)(U_0-3)}. \tag{63}$$

The solitary solutions to (60) are illustrated in Fig. 3. Note that there exist two types of solutions: the bright solitary solution is obtained for $U_0 > 3$ (Fig. 3 (a)); the solitary solution has two singularities for $U_0 < 1$ (Fig. 3 (b)).

The same numerical experiment as in Section 6.1 is performed for (60). The error between the approximate numerical solution to (60) and the analytical solitary solution to (62) is illustrated in Fig. 4. Note that the error is almost equal to zero on the curve defined by (63).

7. Concluding remarks

The correct kink and bright/dark solitary solutions to the equations considered in [1] have been derived. Existence conditions in the space of initial conditions and the equation parameters have been obtained. The results are verified by numerical experiments.

Our exposition of [1] implies that the exp-function method for the construction of exact solitary solutions to fractional-order nonlinear differential equations belongs to the same plethora of recently developed techniques that have attracted a significant amount of criticism. The main deficiencies of this method are discussed at length in [7–13] – failure to verify obtained solutions and even to consider initial conditions and equation parameters for which the solutions hold true yields the incorrect and misleading results of [1].

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