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Special Multiplicative Operators for the Solution of ODE – Invariants and Representations

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Abstract

The generalized multiplicative operator of differentiation is introduced in this paper. It is shown that the generalized multiplicative operator can be expressed as a product of two noncommutative but multiplicative exponential operators, though the generalized multiplicative operator is not an exponential operator itself. The generalized multiplicative operator is effectively exploited for the construction of solutions to nonlinear ordinary differential equations through formal transformations of invariants and representations of initial conditions. The concept of the generalized multiplicative operator provides the insight into the algebraic structure of solutions to nonlinear ordinary differential equations which cannot be identified using conventional exponential operators.

Keywords: ordinary differential equation, multiplicative operator, invariant

1. Introduction

The construction of analytic solutions to nonlinear ordinary differential equations (ODE) is an important research topic. Numerous techniques have been developed for that purpose during the last decades. An explicit algorithm based on the Laurent series for the construction of meromorphic solutions of autonomous nonlinear ODE is presented in [1]. The frequency domain approach is used to prove the existence of a unique bounded, exponentially stable solution to some third order nonlinear differential equations [2]. Existence and boundary behavior for singular nonlinear ODE is investigated in [3]. WTC-Kruskal algorithm is developed in [4] in order to study the Painleve property of nonlinear ODE. Differential transform method has been successfully exploited for solving nonlinear ODE and their systems [5, 6]. The Adomian decomposition method is used to construct the solution in a form of an infinite series where the components are usually determined recurrently [7]. Semi-analytical Chebyshev collocation method is used to solve high-order nonlinear ODE in [8]. We list only a small fraction of different techniques and semi-analytical algorithms for the construction of exact solutions to ODE.

The application of algebraic techniques for the construction of analytic solutions to ODE is a classical field of research [9, 10]. An overview on developments of algebraic theory approach to ODE is given in [11]. The application of algebraic theory to the numerical treatment of ODE is studied in [12]. The differential operator is one of the key concepts in the algebraic theory of differential equations [13]. The exponential differential operator is especially useful for these purposes [14, 15].

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It is well known that the concept of the invariant plays an important role in mathematics in general [16]. In particular, special invariants of ODE have been recently formulated in the context of geometrical analysis of differential equations [17, 18]. The main objective of this paper is to introduce the concept of the generalized multiplicative operator of differentiation and to demonstrate its applicability in solving practical problems. Moreover, we will demonstrate the relationship among the generalized multiplicative operator of differentiation and invariants of differential equations what will help to develop special techniques for the construction of analytic solutions to nonlinear ODE problems.

This paper is organized as follows. Symbols and notations are listed in section 2; the generalized multiplicative operator is introduced in section 3; the expression of the solution in the operator form is derived in section 4; a number of examples are given in section 5 and concluding remarks are given in the last section.

2. Symbols and notations

The following notations will be used throughout the manuscript (appropriate definitions will be given later):

n – the order of the explicit ordinary differential equation;

y – the dependent variable;

$s_0, s_1, \ldots, s_{n-1}$ – Cauchy parameters (initial conditions);

κ – the canonical variable (the center of the series expansion of the solution);

x – the free variable;

$p(\kappa, s_0, s_1, \ldots, s_{n-1})$ – an **R**-valued function of κ and Cauchy parameters;

$f(x, \kappa, s_0, s_1, \ldots, s_{n-1})$ – an **R**-valued function of x, κ and Cauchy parameters;

$\Phi_{\kappa, s_0, s_1, \ldots, s_{n-1}}$ – the set of functions $p(\kappa, s_0, s_1, \ldots, s_{n-1})$;

$\Phi_{x, \kappa, s_0, s_1, \ldots, s_{n-1}}$ – the set of functions $f(x, \kappa, s_0, s_1, \ldots, s_{n-1})$;

s, t – Cauchy parameters for $n = 2$ (i.e. $s := s_0$; $t := s_1$);

D_x, D_κ, D_{s_0}, …, $D_{s_{n-1}}$ – ordinary differential operators in respect of variables x, κ, s_0, …, s_{n-1};

D_y – the generalized differential operator;

M, M_0 – multiplicative operators;

G – the generalized multiplicative operator;

$v(\kappa, s_0, s_1, \ldots, s_{n-1})$ – the invariant associated with D_y;

3. The generalized multiplicative operator

3.1 Existing designs (or original designs)

Functions of two types are used in this paper. Functions of the first type $p_j = p_j(\kappa, s_0, \ldots, s_{n-1})$ describe the mapping

$p_j : \mathrm{I}_\kappa \times \mathrm{I}_{s_0} \times \cdots \times \mathrm{I}_{s_{n-1}} \to \boldsymbol{R}$; where $\mathrm{I}_\kappa, \mathrm{I}_{s_0}, \ldots, \mathrm{I}_{s_{n-1}} \subset \boldsymbol{R}$ are variation intervals (or unions of intervals) of variables $\kappa, s_0, \ldots, s_{n-1} \in \boldsymbol{R}$. These functions are differentiable any number times in respect of every variable. It can be noted that the identification of variation intervals (or unions of intervals) is a straightforward task whenever the expression of $p_j(\kappa, s_0, \ldots, s_{n-1})$ is given explicitly. For example, the function

$$p(\kappa, s_0) = \frac{1}{\kappa\left(1+\sqrt{1-4s_0}\right)} \tag{1}$$

is defined and differentiable any number of times in respect of κ and s_0 when $\kappa \in (-\infty;0) \cup (0;+\infty)$ and $s_0 \in \left(-\infty; \frac{1}{4}\right)$ (the principal square root is considered in Eq. (1)). The analysis of functions of the first type is not the objective of this paper, but we will consider only such functions $p_j = p_j(\kappa, s_0, \ldots, s_{n-1})$ that intervals $\mathrm{I}_\kappa, \mathrm{I}_{s_0}, \ldots, \mathrm{I}_{s_{n-1}}$ exist and are not empty sets. The set of functions of the first type is denoted as $\Phi_{\kappa, s_0, s_1, \ldots, s_{n-1}}$.

Functions of the second type are constructed from functions of the first type using the following algorithm.

(i) Construct the power series:

$$f_0(x, \kappa, s_0, \ldots, s_{n-1}) = \sum_{j=0}^{+\infty} \frac{x^j}{j!} p_j(\kappa, s_0, \ldots, s_{n-1}) \tag{2}$$

(ii) Extend the function $f_0(x, \kappa, s_0, \ldots, s_{n-1})$ to a wider domain (if it is possible) using classical extension techniques. The extended function $f(x, \kappa, s_0, \ldots, s_{n-1})$ is denoted as the second type function.

For example, the series

$$f_0(x, \kappa, s_0) = \sum_{j=0}^{+\infty} \frac{x^j}{j!} \left(j! \left(\frac{1}{\kappa\left(1+\sqrt{1-4s_0}\right)} \right)^j \right) = \sum_{j=0}^{+\infty} \left(\frac{x}{\kappa\left(1+\sqrt{1-4s_0}\right)} \right)^j$$

can be extended to a function

$$f(x, \kappa, s_0) = \frac{\kappa\left(1+\sqrt{1-4s_0}\right)}{\kappa\left(1+\sqrt{1-4s_0}\right) - x}$$

for $s_0 \in (-\infty;0)$ and $\kappa\left(1+\sqrt{1-4s_0}\right) \neq x$. From now on we will use the equality

$$\sum_{j=0}^{+\infty} \frac{x^j}{j!} \left(j! \left(\frac{1}{\kappa\left(1+\sqrt{1-4s_0}\right)} \right)^j \right) = \frac{\kappa\left(1+\sqrt{1-4s_0}\right)}{\kappa\left(1+\sqrt{1-4s_0}\right) - x}$$

assuming that the transformation into the extended function does not cause any misunderstandings and will not specify the domain of x, κ and s_0.

Other forms of the second type functions can be used. Typical cases (structures of f_0) are listed below:

$$f_0(x, \kappa, s_0, \ldots, s_{n-1}) = \sum_{j=0}^{+\infty} \frac{(x-\kappa)^j}{j!} p_j(\kappa, s_0, \ldots, s_{n-1});$$

$$f_0(\kappa, s_0, \ldots, s_{n-1}) = \sum_{j=0}^{+\infty} \frac{\kappa^j}{j!} p_j(\kappa, s_0, \ldots, s_{n-1}).$$

It can be noted that it is not necessary to introduce the function norm (neither for the first type functions nor for the second type functions) in the process of the construction of analytic solutions of nonlinear ordinary differential equations.

The set of extended functions is denoted as $\Phi_{x,\kappa,s_0,\ldots,s_{n-1}}$ ($\Phi_{\kappa,s_0,\ldots,s_{n-1}} \subset \Phi_{x,\kappa,s_0,\ldots,s_{n-1}}$).

3.2 The generalized operator of differentiation

Let us consider an explicit ODE:

$$\frac{d^n y}{dx^n} = P_n\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^{n-1}y}{dx^{n-1}}\right); \quad (3)$$

with initial conditions

$$\left.\frac{d^k y(x, \kappa, s_0, \ldots, s_{n-1})}{dx^k}\right|_{x=\kappa} = s_k;\ k = 0, 1, \ldots, n-1; \quad (4)$$

where $y = y(x, \kappa, s_0, \ldots, s_{n-1})$ is the solution to the initial value problem (Eq. (3, 4)); $n \in \mathbf{N}$ is the fixed order of the differential equation; $P_n = P_n(\kappa, s_0, \ldots, s_{n-1})$ is a function of the first type. Then, the generalized operator of differentiation D_y associated with Eq. (1) reads [19]:

$$D_y := D_\kappa + s_1 D_{s_0} + s_2 D_{s_1} + \cdots + s_{n-1} D_{s_{n-2}} + P_n(\kappa, s_0, \ldots, s_{n-1}) D_{s_{n-1}} \quad (5)$$

This paper is organized as follows. Symbols and notations are enlisted in section 2; the generalized multiplicative operator is introduced in section 3; the expression of the solution in the operator form is derived in section 4; a number of examples are given in section 5 and concluding remarks are given in the last section.

Conventional properties of differentiation hold for D_y. Several properties are listed below:

(i) $D_y(c_1 f_1 + c_2 f_2) = c_1 D_y f_1 + c_2 D_y f_2$; where $c_1, c_2 \in \boldsymbol{R}$; $f_1, f_2 \in \Phi_{x,\kappa,s_0,\ldots,s_{n-1}}$.

(ii) $D_y(f_1 \cdot f_2) = (D_y f_1) f_2 + f_1 (D_y f_2)$.

(iii) $D_y \dfrac{f_1}{f_2} = \dfrac{(Df_1) f_2 - f_1 (Df_2)}{(f_2)^2}$.

(iv) $D_y^m(f_1 \cdot f_2) = \sum_{j=0}^{m} \binom{m}{j} (D_y^j f_1) \cdot (D_y^{m-j} f_2)$; where $\binom{m}{j} = \begin{cases} 0, & j > m; \\ \dfrac{m!}{j!(m-j)!}, & j \le m; \end{cases}$ $j, m = 0, 1, 2, \ldots$.

(v) Let $f(z)$ be a function differentiable any number of times in respect of the variable z. If $F(x, \kappa, s_0, \ldots, s_{n-1}) := f(f_1(x, \kappa, s_0, \ldots, s_{n-1}))$ then $D_y F = (D_z f(z))\big|_{z = f_1(x,\kappa,s_0,\ldots,s_{n-1})} \cdot D_y f_1(x, \kappa, s_0, \ldots, s_{n-1})$.

We will prove Property (ii).

Proof.

Without the loss of generality we will prove Property (ii) for $n = 2$. Let us assume that the generalized operator of differentiation reads $D_y = P(s,t)D_s + Q(s,t)D_t$ and functions $P = P(s,t)$; $Q = Q(s,t)$; $f_1 = f_1(s,t)$; $f_2 = f_2(s,t)$ are differentiable in respect of variables *s* and *t* any number of times. Then,

$$D_y(f_1 \cdot f_2) = (PD_s + QD_t)(f_1 \cdot f_2) = P((D_s f_1)f_2 + f_1(D_s f_2)) + Q((D_t f_1)f_2 + f_1(D_t f_2)) =$$
$$= (P(D_s f_1)f_2 + Q(D_t f_1)f_2) + (f_1 P(D_s f_2) + f_1 Q(D_t f_2)) = (D_y f_1)f_2 + f_1(D_y f_2).$$

End of proof.

Other properties can be proved analogously.

3.3 The multiplicative operator

D_y can be exploited to construct the multiplicative operator *M*:

$$M := \sum_{j=0}^{+\infty} \frac{x^j}{j!} D_y^j . \quad (6)$$

The operator *M* satisfies following equalities [19, 20]:

(i) $M(a_1 f_1 + a_2 f_2) = a_1 M f_1 + a_2 M f_2$; $a_1, a_2 \in \mathbf{R}$; $f_1, f_2 \in \Phi_{\kappa, s_0, \ldots, s_{n-1}}$.

(ii) $M\kappa^m = (\kappa + x)^m$; $m \in \mathbf{Z}_0$.

(iii) $Mf_1(\kappa, s_0, \ldots, s_{n-1}) = f_1(\kappa + x, Ms_0, \ldots, Ms_{n-1})$.

(iv) $M \dfrac{f_1(\kappa, s_0, \ldots, s_{n-1})}{f_2(\kappa, s_0, \ldots, s_{n-1})} = \dfrac{f_1(\kappa + x, Ms_0, \ldots, Ms_{n-1})}{f_2(\kappa + x, Ms_0, \ldots, Ms_{n-1})}$.

Without the loss of generality we will prove the equality $Mf_1(s,t) = f_1(Ms, Mt)$ when $D_y = P(s,t)D_s + Q(s,t)D_t$.

Proof.

Let $y_1 := Ms = y_1(x,s,t)$; $y_2 := Mt = y_2(x,s,t)$; $z := Mf_1(s,t) = z(x,s,t)$ and $w := f_1(Ms, Mt) = f_1(y_1(x,s,t), y_2(x,s,t))$. Then,

$$D_x M f_1(s,t) = \frac{\partial}{\partial x}\left(\sum_{j=0}^{+\infty} \frac{x^j}{j!} D_y^j f_1(s,t)\right) = \sum_{j=1}^{+\infty} \frac{x^{j-1}}{(j-1)!} D_y^j f_1(s,t) = D_y \sum_{j=0}^{+\infty} \frac{x^j}{j!} D_y^j f_1(s,t) = D_y M f_1(s,t) =$$
$$= PD_s M f_1(s,t) + QD_t M f_1(s,t).$$

The last equality yields the following differential equation with partial derivatives:

$$\frac{\partial z}{\partial x} - P\frac{\partial z}{\partial s} - Q\frac{\partial z}{\partial t} = 0; \quad (7)$$

where $z(0,s,t) = f_1(s,t)$. Analogously,

$$\frac{\partial y_1}{\partial x} - P\frac{\partial y_1}{\partial s} - Q\frac{\partial y_1}{\partial t} = 0\,;\;\; y_1(0,s,t) = s\,; \tag{8}$$

and

$$\frac{\partial y_2}{\partial x} - P\frac{\partial y_2}{\partial s} - Q\frac{\partial y_2}{\partial t} = 0\,;\;\; y_2(0,s,t) = t\,. \tag{9}$$

Now, it can be observed that:

$$D_x w = \frac{\partial f_1(y_1,y_2)}{\partial x} = \left.\frac{\partial f_1(u,v)}{\partial u}\right|_{\substack{u=y_1(x,s,t)\\ v=y_2(x,s,t)}}\left(P\frac{\partial y_1}{\partial s} + Q\frac{\partial y_1}{\partial t}\right) + \left.\frac{\partial f_1(u,v)}{\partial v}\right|_{\substack{u=y_1(x,s,t)\\ v=y_2(x,s,t)}}\left(P\frac{\partial y_2}{\partial s} + Q\frac{\partial y_2}{\partial t}\right) =$$

$$= P\left(\left.\frac{\partial f_1(u,v)}{\partial u}\right|_{\substack{u=y_1(x,s,t)\\ v=y_2(x,s,t)}}\cdot\frac{\partial y_1}{\partial s} + \left.\frac{\partial f_1(u,v)}{\partial v}\right|_{\substack{u=y_1(x,s,t)\\ v=y_2(x,s,t)}}\cdot\frac{\partial y_2}{\partial s}\right) +$$

$$+ Q\left(\left.\frac{\partial f_1(u,v)}{\partial u}\right|_{\substack{u=y_1(x,s,t)\\ v=y_2(x,s,t)}}\cdot\frac{\partial y_1}{\partial t} + \left.\frac{\partial f_1(u,v)}{\partial v}\right|_{\substack{u=y_1(x,s,t)\\ v=y_2(x,s,t)}}\cdot\frac{\partial y_2}{\partial t}\right) = P\frac{\partial w}{\partial s} + Q\frac{\partial w}{\partial t}.$$

The last relationship yields the equality:

$$\frac{\partial w}{\partial x} - P\frac{\partial w}{\partial s} - Q\frac{\partial w}{\partial t} = 0\,;\;\; w(0,s,t) = f\left(y_1(0,s,t), y_2(0,s,t)\right) = f(s,t). \tag{10}$$

Therefore, finally:

$$z(x,s,t) = w(x,s,t).$$

End of proof.

Other equalities can be proved analogously.

It is worth noting that the multiplicative operator defined by Eq. (6) can be considered as the exponential operator:

$$M = \exp(xD_y). \tag{11}$$

Now, let us introduce the operator:

$$M_0 := \sum_{j=0}^{+\infty}\frac{(-\kappa)^j D_x^j}{j!}. \tag{12}$$

Note, that

$$M_0 x^n = \sum_{r=0}^{n}\binom{r}{n}(-\kappa)^r x^{n-r} = (x-\kappa)^n\,;\;\; n = 0,1,2,\ldots.$$

Moreover,

$$M_0 f(x,\kappa,s_0,\ldots,s_{n-1}) = f(x-\kappa,\kappa,s_0,\ldots,s_{n-1}).$$

Thus, the operator M_0 is a multiplicative and an exponential operator at the same time.

3.4 The generalized operator of differentiation

We introduce the generalized multiplicative operator:

$$G := \sum_{j=0}^{+\infty} \frac{(x-\kappa)^j}{j!} D_y^j \,. \tag{13}$$

The following properties hold true:

(i) $G(a_1 f_1 + a_2 f_2) = a_1 G f_1 + a_2 G f_2$; $a_1, a_2 \in \mathbf{R}$; $f_1, f_2 \in \Phi_{\kappa, s_0, \ldots, s_{n-1}}$.

(ii) $G\kappa^m = x^m$; $m \in \boldsymbol{Z}_0$.

(iii) $G f_1(\kappa, s_0, \ldots, s_{n-1}) = f_1(x, Gs_0, \ldots, Gs_{n-1})$.

(iv) $G \dfrac{f_1(\kappa, s_0, \ldots, s_{n-1})}{f_2(\kappa, s_0, \ldots, s_{n-1})} = \dfrac{f_1(x, Gs_0, \ldots, Gs_{n-1})}{f_2(x, Gs_0, \ldots, Gs_{n-1})}$.

(v) Let $Ms_k = g_k(x, \kappa, s_0, \ldots, s_{n-1})$. Then $Gs_k = g_k(x-\kappa, \kappa, s_0, \ldots, s_{n-1})$; $k = 0, 1, \ldots, n-1$.

We will prove the third equality (other equalities can be proved analogously).

Proof.

The second property of the multiplicative operator yields:

$$M f_1(\kappa, s_0, \ldots, s_{n-1}) = f_1\left(x+\kappa, \sum_{j=0}^{+\infty} \frac{x^j}{j!}\left(D_y^j s_0\right), \ldots, \sum_{j=0}^{+\infty} \frac{x^j}{j!}\left(D_y^j s_{n-1}\right)\right).$$

The replacement of the variable *x* by the expression $x-\kappa$ (what is possible) yields:

$$\sum_{j=0}^{+\infty} \frac{(x-\kappa)^j}{j!}\left(D_y^j f_1(\kappa, s_0, \ldots, s_{n-1})\right) = f_1\left(x-\kappa+\kappa, \sum_{j=0}^{+\infty} \frac{(x-\kappa)^j}{j!}\left(D_y^j s_0\right), \ldots, \sum_{j=0}^{+\infty} \frac{(x-\kappa)^j}{j!}\left(D_y^j s_{n-1}\right)\right),$$

what concludes the proof.

End of proof.

Definitions and properties of operators *M*, M_0 and *G* (Eq. (11, 12, 13)) yield the following equality.

Corollary 1.

$$G = M_0 \cdot M \,.$$

Thus, the generalized multiplicative operator *G* is a product of two noncommutative but multiplicative and exponential operators. But the operator *G* is not an exponential operator. It is worth noting that exponential operators are widely used in geometric-operator calculus [17, 21, 22]. We will demonstrate that the generalized multiplicative operator *G* can be effectively exploited for the construction of solutions to ODE problems.

4. The expression of the solution to ODE in the operator form

Theorem 1.

The solution to the initial ODE problem Eq. (3, 4) can be expressed in the following form [19]:

$$y = Gs_0 .$$

Without the loss of generality we will prove that the solution to the differential equation

$$\frac{d^2 y}{dx^2} = P_2\left(x, y, \frac{dy}{dx}\right);\; y = y(x,\kappa,s,t);\; y(\kappa,\kappa,s,t) = s\,;\; \left.\frac{dy(x,\kappa,s,t)}{dx}\right|_{x=\kappa} = t \quad (14)$$

reads:

$$y(x,\kappa,s,t) = \sum_{j=0}^{+\infty} \frac{(x-\kappa)^j}{j!}\left(D_\kappa + tD_s + P_2(\kappa,s,t)D_t\right)^j s . \quad (15)$$

Proof.

It is clear that $D_y = D_\kappa + tD_s + P_2(\kappa,s,t)D_t$. The operator D_y satisfies all properties of the generalized operator of differentiation. Now, let $z(x,\kappa,s,t) = \sum_{j=0}^{+\infty} \frac{x^j}{j!} D_y^j s = Ms$. Then,

$$\frac{dz(x,\kappa,s,t)}{dx} = D_x Ms = D_y Ms = MD_y s = Mt .$$

Analogously,

$$\frac{d^2 z(x,\kappa,s,t)}{dx^2} = D_x Mt = D_y Mt = MD_y t = MP_2(\kappa,s,t) = P_2(M\kappa, Ms, Mt) = P_2\left(x+\kappa, z(x,\kappa,s,t), \frac{dz(x,\kappa,s,t)}{dx}\right).$$

Thus,

$$\frac{d^2 z(x-\kappa,\kappa,s,t)}{dx^2} = P_2\left(x, z(x-\kappa,\kappa,s,t), \frac{dz(x-\kappa,\kappa,s,t)}{dx}\right).$$

Therefore,

$$z(x-\kappa,\kappa,s,t) = Gs = y(x,\kappa,s,t) \text{ and } y(\kappa,\kappa,s,t) = s\,;\; \left.\frac{dy(x,\kappa,s,t)}{dx}\right|_{x=\kappa} = t .$$

End of proof.

Eq. (15) can be considered as the generalization of the Picard formula [23] describing the solution of an ordinary differential equation in a power series form.

Corollary 2. The following equality holds:

$$\frac{d^k y(x,\kappa,s_0,\ldots,s_{n-1})}{dx^k} = Gs_k . \tag{16}$$

Proof.

It is clear that $D_y^k s_0 = s_k$ for $k = 0,1,\ldots,n-1$. Then,

$$\frac{d^k y(x,\kappa,s_0,\ldots,s_{n-1})}{dx^k} = D_x^k \sum_{j=0}^{+\infty} \frac{(x-\kappa)^j}{j!} D_y^j s_0 = \sum_{j=k}^{+\infty} \frac{(x-\kappa)^{j-k}}{(j-k)!} D_y^j s_0 = \sum_{j=0}^{+\infty} \frac{(x-\kappa)^j}{j!} D_y^{j+k} s_0 =$$

$$= \sum_{j=0}^{+\infty} \frac{(x-\kappa)^j}{j!} D_y^j s_k = Gs_k .$$

End of Proof.

Properties of multiplicative operators M and G yield the fact that the solution to the ODE initial value problem (Eq. (3, 4)) does satisfy the equality:

$$\frac{\partial y(x+\kappa,\kappa,s_0,\ldots,s_{n-1})}{\partial x} - \frac{\partial y(x+\kappa,\kappa,s_0,\ldots,s_{n-1})}{\partial \kappa} - s_1 \frac{\partial y(x+\kappa,\kappa,s_0,\ldots,s_{n-1})}{\partial s_0} - \cdots - s_{n-1} \frac{\partial y(x+\kappa,\kappa,s_0,\ldots,s_{n-1})}{\partial s_{n-2}} -$$

$$- P_n(\kappa,s_0,\ldots,s_{n-1}) \frac{\partial y(x+\kappa,\kappa,s_0,\ldots,s_{n-1})}{\partial s_{n-1}} = 0.$$

For example, the solution to the initial value problem $\frac{dy}{dx} = y + x$; $y(\kappa,\kappa,s) = s$ takes the form $y(x,\kappa,s) = -1 - x + (s+\kappa+1)\exp(x-\kappa)$. But the function $y(x+\kappa,\kappa,s) = (s+\kappa+1)\exp(x) - (x+\kappa+1) := z(x,\kappa,s)$ with the initial condition $z(0,\kappa,s) = s$ satisfies the following differential equation:

$$\frac{\partial z(x,\kappa,s)}{\partial x} - \frac{\partial z(x,\kappa,s)}{\partial \kappa} - (s+\kappa)\frac{\partial z(x,\kappa,s)}{\partial s} = 0 .$$

4.1 Invariants of the generalized operator of differentiation and their properties

Definition 1. A function $v = v(x,\kappa,s_0,\ldots,s_{n-1})$; $v \in \Phi_{x,\kappa,s_0,\ldots,s_{n-1}}$ is an invariant of D_y if

$$D_y v = 0 . \tag{17}$$

The set of invariants is denoted as $\{v | D_y v = 0\} := \text{Ker } D_y$. Properties of invariants of D_y are listed below.

(i) Let $v_1, v_2 \in \text{Ker } D_y$ and $c_1, c_2 \in \mathbf{C}$. Then $c_1 v_1 + c_2 v_2 ; v_1 v_2 ; \frac{v_1}{v_2} \in \text{Ker } D_y$. The proof follows from properties of D_y.

(ii) Let $\alpha = \alpha(z_1, z_2, \ldots, z_m)$ be a function of variables $z_1, z_2, \ldots, z_m$ and $v_1, v_2, \ldots, v_m \in \text{Ker } D_y$. Then, $\alpha(v_1, v_2, \ldots, v_m) \in \text{Ker } D_y$.

Proof.

$$D_y \alpha(v_1,\ldots,v_m) = \sum_{k=1}^{m} \left. \frac{d\alpha(z_1,\ldots,z_m)}{dz_k} \right|_{z_k = v_k} \cdot D_y v_k = 0 .$$

End of proof.

(iii) Let $v \in \text{Ker } D_y$ and $f \in \Phi_{x,\kappa,s_0,\ldots,s_{n-1}}$. Then, $D_y(vf) = v(D_y f)$; $M(vf) = v(Mf)$; $G(vf) = v(Gf)$.

Proof.

$D_y(vf) = (D_y v)\cdot f + v \cdot D_y f = v(D_y f)$. Other equalities can be proved analogously.

End of proof.

(iv) Let $f \in \Phi_{x,\kappa,s_0,\ldots,s_{n-1}}$. Then the operator

$$\tilde{G}_{\kappa_0} := \sum_{j=0}^{+\infty} \frac{(\kappa - \kappa_0)^j}{j!}(-D_y)^j;\ \kappa_0 \in \boldsymbol{R} \tag{18}$$

yields the invariant $\tilde{G}_{\kappa_0} f \in \text{Ker } D_y$.

Proof.

$$D_y \tilde{G}_{\kappa_0} f = D_y \sum_{j=0}^{+\infty} \frac{(\kappa-\kappa_0)^j}{j!}(-D_y)^j f = \sum_{j=0}^{+\infty} D_y\left(\frac{(\kappa-\kappa_0)^j}{j!}(-D_y)^j f\right) =$$

$$= \sum_{j=0}^{+\infty}\left(\left(D_y \frac{(\kappa-\kappa_0)^j}{j!}\right)\left((-D_y)^j f\right) + \frac{(\kappa-\kappa_0)^j}{j!}\left(-(-D_y)^{j+1} f\right)\right) =$$

$$= \sum_{j=1}^{+\infty}\left(\frac{(\kappa-\kappa_0)^{j-1}}{(j-1)!}(-D_y)^j f\right) - \sum_{j=0}^{+\infty}\frac{(\kappa-\kappa_0)^j}{j!}\left((-D_y)^{j+1} f\right) =$$

$$= \sum_{j=0}^{+\infty}\frac{(\kappa-\kappa_0)^j}{j!}(-D_y)^{j+1} f - \sum_{j=0}^{+\infty}\frac{(\kappa-\kappa_0)^j}{j!}\left((-D_y)^{j+1} f\right) = 0$$

End of proof.

Corollary 3. The replacement of a real complex number κ_0 in the expression of $\tilde{G}_{\kappa_0}$ by a symbol x yields the equality

$$\tilde{G}_x = G; \tag{19}$$

moreover, $Gf \in \text{Ker } D_y$ for $f \in \Phi_{x,\kappa,s_0,\ldots,s_{n-1}}$.

Proof.

$\tilde{G}_x = \sum_{j=0}^{+\infty}\frac{(\kappa-x)^j}{j!}(-D_y)^j = \sum_{j=0}^{+\infty}\frac{(-(x-\kappa))^j}{j!}(-D_y)^j = \sum_{j=0}^{+\infty}\frac{(x-\kappa)^j}{j!}D_y^j = G$. Thus, $\tilde{G}_x f = Gf \in \text{Ker } D_y$ for $f \in \Phi_{x,\kappa,s_0,\ldots,s_{n-1}}$.

End of proof.

Note that the variable x is regarded as a constant in respect of the operator D_y.

Corollary 4.

Eq. (13), Eq. (18) and Eq. (19) yield the equality $D_y y = 0$; therefore

$$y \in \operatorname{Ker} D_y . \tag{20}$$

Corollary 5. Let $f_1, f_2 \in \Phi_{x,\kappa,s_0,\ldots,s_{n-1}}$. Then $Gf_1 = Gf_2$ if and only if $f_1 = f_2$.

Proof.

If $f_1 = f_2$ then $Gf_1 = Gf_2$.

Now, let us assume that $Gf_1 = Gf_2$. Then $G(f_1 - f_2) = \sum_{j=0}^{+\infty} \frac{(x-\kappa)^j}{j!}\left(D_y^j(f_1 - f_2)\right) = 0$ what requires that $D_y^j(f_1 - f_2) = 0$ holds for all $j = 0,1,2,\ldots$. But $D_y^0(f_1 - f_2) = f_1 - f_2$ what concludes the proof.

End of proof.

Corollary 6. Let $f(x,\kappa,s_0,\ldots,s_{n-1}) \in \operatorname{Ker} D_y$. The replacement of the symbol *x* by the symbol κ produces a function $f(\kappa,\kappa,s_0,\ldots,s_{n-1})$. Then, the following equality holds true:

$$Gf(\kappa,\kappa,s_0,\ldots,s_{n-1}) = f(x,\kappa,s_0,\ldots,s_{n-1}). \tag{21}$$

Proof.

The function *f* can be expressed in the form $f(x,\kappa,s_0,\ldots,s_{n-1}) = \sum_{j=0}^{+\infty} v_j \frac{x^j}{j!}$ since $f(x,\kappa,s_0,\ldots,s_{n-1}) \in \operatorname{Ker} D_y$, where $v_j \in \operatorname{Ker} D_y$. But then $f(\kappa,\kappa,s_0,\ldots,s_{n-1}) = \sum_{j=0}^{+\infty} v_j \frac{\kappa^j}{j!}$ what immediately yields Eq. (21).

End of proof.

Let $y(x,\kappa,s,t)$ be the solution to differential equation (14) and $y(x_0,\kappa,s,t) := v_{x_0}(\kappa,s,t) := v$; $\left.\frac{dy(x,\kappa,s,t)}{dx}\right|_{x=x_0} := u_{x_0}(\kappa,s,t) := u$ where $x_0 \in R$ is not a singular point of differential equation (14). Then,

$$\operatorname{Ker} D_y = \left\{ \sum_{k,l \in \mathbf{Z}_0} a_{kl} v^k u^l \,\middle|\, a_{kl} \in C \right\}. \tag{22}$$

It can be noted that Eq. (22) is not the only representation of $\operatorname{Ker} D_y$ in terms of *u* and *v*. In general, other representations (in other terms) do exist.

4.2 The canonical parameter of the generalized operator of differentiation

Definition 2. A function $\chi = \chi(\kappa,s_0,\ldots,s_{n-1})$; $\chi \in \Phi_{\kappa,s_0,\ldots,s_{n-1}}$ is the canonical parameter of D_y if

$$D_y \chi = 1. \tag{23}$$

Corollary 7. The variable κ is a canonical parameter of D_y because $D_y \kappa = 1$. Moreover, all other canonical parameters of D_y can be expressed in the form:

$$\chi = \kappa + v \tag{24}$$

where $v \in \text{Ker } D_y$.

Corollary 8. Let χ_1 and χ_2 be two canonical parameters of D_y. Then,

$$\chi_1 - \chi_2 \in \text{Ker } D_y .$$

4.3 Structural expressions of the solution to ODE

We will consider several typical structural expressions of solutions. Let us assume that the function $z = z(\kappa, s_0, \ldots, s_{n-1})$ can be expressed in the form:

$$z = \varphi(\kappa, v_1, \ldots, v_m) \tag{25}$$

where $\varphi \in \Phi_{\kappa, s_0, \ldots, s_{n-1}}$; $v_j = v_j(\kappa, s_0, \ldots, s_{n-1}) \in \text{Ker } D_y$; $j = 1,2,\ldots,m$. Then, properties of the generalized operator of differentiation yield:

$$Gz = \varphi(x, v_1, \ldots, v_m). \tag{26}$$

Note that $Gv = v + \sum_{j=1}^{+\infty} \frac{(x-\kappa)^j}{j!} D_y^j v = v$. Various particular cases of Eq. (25) could be considered. Several typical examples are listed below.

Theorem 2.

Let us assume that s_0 can be expressed in the form:

$$s_0 = \varphi(\kappa, v_1, \ldots, v_m). \tag{27}$$

Then,

$$y = \varphi(x, v_1, \ldots, v_m). \tag{28}$$

Proof.

The proof follows from Eq. (13) and Eq. (26).

End of proof.

Eq. (28) also represents the structural expression of the solution in Eq. (13).

It appears that if one is able to identify invariants and to construct the expression of s_0 (Eq. (27)) then there is no need to solve the initial problem defined by Eq. (3, 4) – the solution is automatically generated by Eq. (28). Since this result is of fundamental importance, we denote the expression in Eq. (27) as the s_0 – representation.

The natural question arises what is easier – to find invariants or to solve the initial problem using conventional techniques. A discussion on these questions is provided in section 5.

Corollary 9. A particular case of the s_0 – representation:

$$s_0 = \sum_{k=1}^{\infty} v_k \cdot f_k(\kappa), \tag{29}$$

where $v_k = v_k(\kappa, s_0, \ldots, s_{n-1}) \in \operatorname{Ker} D_y$ for all k yields the solution to the ODE defined by Eq. (3, 4):

$$y = \sum_{k=1}^{\infty} v_k \cdot f_k(x). \tag{30}$$

Eq. (30) represents another structural expression of the solution in Eq. (13).

Note that $y = \sum_{k=1}^{\infty} v_k \cdot x^k$ if $f_k(\kappa) = \kappa^k$ (what is a rather common situation).

Corollary 10.

Let us assume that the expression of the invariant $v = v(\kappa, s_0) \in \operatorname{Ker} D_y$ is given. Then $Gv = v(G\kappa, Gs_0)$ what yields the algebraic equation:

$$v = v(x, y(x, \kappa, s_0, \ldots, s_{n-1})). \tag{31}$$

Then the solution y could be expressed from Eq. (31). Thus it is possible (not always) to reduce the initial problem defined by Eq. (3, 4) to the solution of the algebraic problem defined by equation Eq. (31).

It can be noted that other generalizations are also possible.

5. Examples

A number of examples are given in this Section. We start from the most primitive examples and continue with more demanding nonlinear ODE problems.

Example 1. Let us consider a differential equation $\frac{dy}{dx} = 0$; $y = y(x, \kappa, s)$ with the initial condition $y(\kappa, \kappa, s) = s$. Then,

$D_y = D_\kappa + 0 \cdot D_s$; $G = \sum_{j=0}^{+\infty} \frac{(x-\kappa)^j}{j!} D_\kappa^j$. The invariant reads $v(\kappa, s) := s$ because $D_y s = s$. Then, the s-representation reads: $s = s$. Therefore, $Gs = s$; $y(x, \kappa, s) = Gs$. Finally, $y(x, \kappa, s) = s$; $y(\kappa, \kappa, s) = s$ and $\operatorname{Ker} D_y = \left\{ \sum_{k \in Z_0} a_k s^k \middle| a_k \in \boldsymbol{R} \right\}$.

Example 2. Let us consider a differential equation $\frac{d^2y}{dx^2}=0$; $y=y(x,\kappa,s,t)$ with initial conditions $y(c;c,s,t)=s$; $\left.\frac{dy(x,\kappa,s,t)}{dx}\right|_{x=\kappa}=t$. Then, $D_y=D_\kappa+tD_s+0D_t$. The invariant reads: $v(\kappa,s,t)=s-\kappa t$ because $(D_\kappa+tD_s)(s-\kappa t)=-t+t=0$. Then the s-representation reads: $s=(s-\kappa t)+\kappa t$. Now, $Gs=(s-\kappa t)+G\kappa t$ (note that $G\kappa t=(G\kappa)(Gt)=xt$). Thus $y=s-\kappa t+xt$; $y(\kappa,\kappa,s,t)=s$; $\left.\frac{dy(x,\kappa,s,t)}{dx}\right|_{x=\kappa}=t$. Note that $Gt=\frac{dy(x,\kappa,s,t)}{dx}=t$ and

$$\text{Ker } D_y=\left\{\sum_{k,l\in Z_0}a_{kl}(s-\kappa t+x_0t)^k t^l \,\middle|\, a_{kl},x_0\in \boldsymbol{R}; x_0\neq 0\right\}.$$

Example 3. Let us consider a differential equation $\frac{dy}{dx}=P_1(x)$; $y=y(x,\kappa,s)$ with the initial condition $y(\kappa,\kappa,s)=s$. The generalized operator of differentiation reads: $D_y=D_\kappa+P_1(\kappa)D_s$; $G=\sum_{j=0}^{+\infty}\frac{(x-\kappa)^j}{j!}(D_\kappa+P_1(\kappa)D_s)$. Now, let us define a primitive function $\hat{P}_1(x)$ for $P_1(x)$: $\hat{P}_1(\kappa)=\int_\eta^\kappa P_1(z)dz$; $\hat{P}_1(\eta)=0$ where $\eta\in\boldsymbol{R}$ and $\hat{P}_1(\kappa)$ exists. Then, the invariant reads: $v(\kappa,s)=s-\hat{P}_1(\kappa)$ because $(D_\kappa+P_1(\kappa)D_s)(s-\hat{P}_1(\kappa))=P_1(\kappa)-P_1(\kappa)=0$. Then the s-representation reads: $s=(s-\hat{P}_1(\kappa))+\hat{P}_1(\kappa)$. Thus, $Gs=(s-\hat{P}_1(\kappa))+\hat{P}_1(x)$. Finally $y(x,\kappa,s)=(s-\hat{P}_1(\kappa))+\hat{P}_1(x)$; $y(\kappa,\kappa,s)=s$ and

$$\text{Ker } D_y=\left\{\sum_{k\in Z_0}a_k\left(s-\hat{P}_1(\kappa)+\hat{P}_1(x_0)\right)^k \,\middle|\, x_0,a_k\in\boldsymbol{R}; \left|\hat{P}_1(x_0)\right|<+\infty\right\}.$$

Example 4. Let us consider a differential equation $\frac{dy}{dx}=\frac{P_1(x)}{Q_1(y)}$; $y=y(x,\kappa,s)$ with the initial condition $y(\kappa,\kappa,s)=s$. Then, $D_y=D_\kappa+\frac{P_1(\kappa)}{Q_1(s)}D_s$; $G=\sum_{j=0}^{+\infty}\frac{(x-\kappa)^j}{j!}\left(D_\kappa+\frac{P_1(\kappa)}{Q_1(s)}D_s\right)^j$. Let $\hat{P}_1(x)$ and $\hat{Q}_1(x)$ be primitive functions for $P_1(x)$ and $Q_1(x)$. Then, the invariant reads: $v(\kappa,s)=\hat{Q}_1(s)-\hat{P}_1(\kappa)$, because $\left(D_\kappa+\frac{P_1(\kappa)}{Q_1(s)}D_s\right)(\hat{Q}_1(s)-\hat{P}_1(\kappa))=-P_1(\kappa)+P_1(\kappa)=0$. Now, the s-representation can be expressed in the implicit form $\hat{Q}_1(s)=(\hat{Q}_1(s)-\hat{P}_1(\kappa))+\hat{P}_1(\kappa)$. Therefore, $G\hat{Q}_1(s)=(\hat{Q}_1(s)-\hat{P}_1(\kappa))+G\hat{P}_1(\kappa)$; $\hat{Q}_1(G(s))=(\hat{Q}_1(s)-\hat{P}_1(\kappa))+\hat{P}_1(G(\kappa))$; $\hat{Q}_1(y(x,\kappa,s))=\hat{Q}_1(s)-\hat{P}_1(\kappa)+\hat{P}_1(x)$. Finally $y(x,\kappa,s)=\hat{Q}_1^{-1}(\hat{Q}_1(s)-\hat{P}_1(\kappa)+\hat{P}_1(x))$. The explicit solution exists if the inverse function $\hat{Q}_1^{-1}(x)$ does exist.

Example 5. Let us consider a differential equation $\frac{dy}{dx}=y^2$; $y=y(x,\kappa,s)$ with the initial condition $y(\kappa,\kappa,s)=s$. Then, $D_y=D_\kappa+s^2D_s$. It can be observed that $v=v(\kappa,s)=\frac{s}{1+s\kappa}$ because

$$D_yv=(D_\kappa+s^2D_s)\frac{s}{1+s\kappa}=-\frac{s^2}{(1+s\kappa)^2}+s^2\frac{(1+s\kappa)-s\kappa}{(1+s\kappa)^2}=0.$$

Thus, $\frac{s}{1+s\kappa}=G\frac{s}{1+s\kappa}=\frac{Gs}{1+(Gs)(G\kappa)}=\frac{y(x,\kappa,s)}{1+xy(x,\kappa,s)}$. The algebraic equation for the identification of y takes the form: $\frac{s}{1+s\kappa}=\frac{y}{1+xy}$. Finally $y(x,\kappa,s)=\frac{s}{1-s(x-\kappa)}$. It can be noted that $D_y y(x,\kappa,s)=\left(D_\kappa+s^2D_s\right)\frac{s}{1-s(x-\kappa)}=0$ and $y(\kappa,\kappa,s)=s$ and $\text{Ker } D_y=\left\{\sum_{k\in Z_0} a_k\left(\frac{s}{1-s(x_0-\kappa)}\right)^k \middle| a_k, x_0\in \boldsymbol{R}\right\}$.

Example 6. Let us consider a linear ordinary differential equation $\frac{d^2y}{dx^2}+a\frac{dy}{dx}+by=0$; $y=y(x,\kappa,s,t)$ with initial conditions $y(\kappa,\kappa,s,t)=s$; $\left.\frac{dy(x,\kappa,s,t)}{dx}\right|_{x=\kappa}=t$. Then, $D_y=D_\kappa+tD_s-(at+bs)D_t$ and invariants become $v_1(\kappa,s,t)=(t-\lambda_2 s)\exp(-\lambda_1\kappa)$; $v_2(\kappa,s,t)=(t-\lambda_1 s)\exp(-\lambda_2\kappa)$ where λ_1 and λ_2 are two different roots of the algebraic equation $\lambda^2+a\lambda+b=0$. Note that $D_y v_1=D_y v_2=0$. Then, the s-representation reads:

$$s=\frac{t-\lambda_2 s}{\lambda_1-\lambda_2}\exp(-\lambda_1\kappa)\exp(\lambda_1\kappa)+\frac{t-\lambda_1 s}{\lambda_2-\lambda_1}\exp(-\lambda_2\kappa)\exp(\lambda_2\kappa).$$

Thus, the solution reads:

$$y=\frac{t-\lambda_2 s}{\lambda_1-\lambda_2}\exp(\lambda_1(x-\kappa))+\frac{\lambda_1 s-t}{\lambda_1-\lambda_2}\exp(\lambda_2(x-\kappa))$$

and

$$\text{Ker } D_y=\left\{\sum_{k,l=0}^{+\infty} a_{kl}\left((t-\lambda_2 s)\exp(-\lambda_1\kappa)\right)^k\cdot\left((t-\lambda_1 s)\exp(-\lambda_2\kappa)\right)^l \middle| a_{kl}\in\boldsymbol{R}\right\}.$$

Example 7. Let us consider a linear ordinary differential equation $\frac{d^2y}{dx^2}-2\lambda_0\frac{dy}{dx}+\lambda_0^2 y=0$; $y=y(x,\kappa,s,t)$ with initial conditions $y(\kappa,\kappa,s,t)=s$; $\left.\frac{dy(x,\kappa,s,t)}{dx}\right|_{x=\kappa}=t$. Then, $D_y=D_\kappa+tD_s+\left(2\lambda_0 t-\lambda_0^2 s\right)D_t$ and invariants read $v_1(\kappa,s,t)=(s-(t-\lambda_0 s))\kappa\exp(-\lambda_0\kappa)$; $v_2(\kappa,s,t)=(t-\lambda_0 s)\exp(-\lambda_0\kappa)$; $D_y v_1=D_y v_2=0$. The s-representation takes the form: $s=((s-(t-\lambda_0 s)\kappa)\exp(-\lambda_0\kappa))\exp(\lambda_0\kappa)+((t-\lambda_0 s)\exp(-\lambda_0\kappa))\exp(\lambda_0\kappa)$.

Thus, the solution reads: $y=(s+(t-\lambda_0 s)(x-\kappa))\exp(\lambda_0(x-\kappa))$ and

$$\text{Ker } D_y=\left\{\sum_{k,l=0}^{+\infty} a_{kl}\left((s-(t-\lambda_0 s))\kappa\exp(-\lambda_0\kappa)\right)^k\cdot\left((t-\lambda_0 s)\exp(-\lambda_0\kappa)\right)^l \middle| a_{kl}\in R\right\}.$$

So far, the solution of trivial differential equations has been demonstrated in Examples 1 – 7. These examples were used to illustrate the specific features of the proposed solution technique. A more demanding problem is investigated in Example 8.

Example 8 (The Riccati type equation). Let us consider a differential equation $\frac{dy}{dx}=\frac{(y-y_1)(y-y_2)}{(y_1-y_2)x}$; $y=y(x,\kappa,s)$ with the initial condition $y(\kappa,\kappa,s)=s$; where $y_1,y_2\in\boldsymbol{C}$; $y_1\neq y_2$. Then,

$D_y=D_\kappa+\frac{(s-y_1)(s-y_2)}{(y_1-y_2)\kappa}\cdot D_s$; $G=\sum_{j=0}^{+\infty}\frac{(x-\kappa)^j}{j!}\left(D_\kappa+\frac{(s-y_1)(s-y_2)}{(y_1-y_2)\kappa}\cdot D_s\right)^j$. Two invariants are appropriate for the *s*-representation: $v_1(\kappa,s)=y_2$; $v_2(\kappa,s)=\frac{s-y_2}{s-y_1}\kappa$ because $D_y y_1=D_y y_2=0$. Really,

$D_y v_2=\frac{s-y_2}{s-y_1}+\frac{(s-y_1)(s-y_2)\kappa}{(y_1-y_2)\kappa}\cdot\frac{(s-y_1)-(s-y_2)}{(s-y_1)^2}=\frac{s-y_2}{s-y_1}\left(1+\frac{y_2-y_1}{y_1-y_2}\right)=0$. Then, the *s*-representation can be expressed in the implicit form: $s=y_2+\frac{(s-y_2)\kappa}{s-y_1}\cdot\frac{s-y_1}{\kappa}$.

Therefore, $Gs=y_2+\frac{(s-y_2)\kappa}{s-y_1}\cdot\frac{Gs-Gy_1}{G\kappa}$; $y(x,\kappa,s)=y_2+\frac{(s-y_2)\kappa}{s-y_1}\cdot\frac{y(x,\kappa,s)-y_1}{x}$; and finally,

$y(x,\kappa,s)=\frac{y_2(s-y_1)x-y_1(s-y_2)\kappa}{(s-y_1)x-(s-y_2)\kappa}$; $y(\kappa,\kappa,s)=s$. The kernel reads:

$\operatorname{Ker} D_y=\left\{\sum_{k\in Z_0}a_k\left(\frac{y_2(s-y_1)x_0-y_1(s-y_2)\kappa}{(s-y_1)x_0-(s-y_2)\kappa}\right)^k\middle| x_0,a_k\in\boldsymbol{R};x_0\neq 0\right\}$ because $x_0=0$ is the singular point of this differential equation.

It can be noted that the same problem can be solved using different invariants (we will exploit a similar invariant to the one used in Example 4): $v(\kappa,s)=\hat{Q}_1(s)-\hat{P}_1(\kappa)$ where $P_1(x)=\frac{1}{(y_1-y_2)x}$; $\hat{P}_1(x)=\frac{\ln x}{y_1-y_2}$; $\hat{P}_1(1)=0$ and $Q_1(y)=\frac{1}{(y-y_1)(y-y_2)}$; $\hat{Q}_1(y)=\frac{1}{y_1-y_2}\ln\frac{y-y_1}{y-y_2}$. Then, the implicit *s*-representation reads:

$\frac{1}{y_1-y_2}\ln\frac{s-y_1}{s-y_2}=\frac{1}{y_1-y_2}\ln\frac{s-y_1}{s-y_2}-\frac{1}{y_1-y_2}\ln\kappa+\frac{1}{y_1-y_2}\ln\kappa$. Then, $G\ln\frac{s-y_1}{s-y_2}=\ln\frac{s-y_1}{s-y_2}-\ln\kappa+G\ln\kappa$ because $D_y\left(\ln\frac{s-y_1}{s-y_2}-\ln\kappa\right)=0$. Therefore, $\ln\frac{y(x,\kappa,s)-y_1}{y(x,\kappa,s)-y_2}=\ln\frac{s-y_1}{s-y_2}-\ln\kappa+\ln x$. Finally, the solution in the explicit form reads: $y(x,\kappa,s)=\frac{y_2(s-y_1)x-y_1(s-y_2)\kappa}{(s-y_1)x-(s-y_2)\kappa}$; $y(\kappa,\kappa,s)=s$.

An even more complex problem is investigated in Example 9.

Example 9 (Invariants and the s-representation of the Liouville type equation). Let us consider the differential equation

$$\frac{dy}{dx}=-\frac{y}{x}\sqrt{1-4\gamma y};\ \gamma\in R;\ y=y(x,\kappa,s), \tag{32}$$

with the initial condition $y(\kappa,\kappa,s)=s$. Then, $D_y=D_\kappa-\frac{s}{\kappa}\sqrt{1-4\gamma s}D_s$.

The invariant reads: $v(\kappa, s) = \dfrac{s}{\kappa\left(1 - 2\gamma s + \sqrt{1 - 4\gamma s}\right)}$. Really, it is easy to check that

$$\left(D_c - \frac{s}{\kappa}\sqrt{1 - 4\gamma s}D_s\right)\frac{s}{\kappa\left(1 - 2\gamma s + \sqrt{1 - 4\gamma s}\right)} = 0 .$$

Nevertheless, it is clear that finding the invariant can be as much complex problem as solving the original differential equation. It was rather easy to determine invariants in Example 8. But the identification of $v(\kappa, s)$ becomes a difficult task now. In general, one needs to have some sort of algorithm for the construction invariants (especially if differential equations are complex). We are going to present a detailed description of this algorithm in the second part of this paper (the object of the first part is to derive the general framework for the solution of nonlinear ordinary differential equations).

At this point we will illustrate the duality of the problem. One can derive invariants and then the construction of the solution becomes a straightforward task. On the contrary, one can reconstruct invariants if the explicit expression of the solution is available. Note that the algorithm for the construction of invariants (the objective of the second part of the study) does not require the analytic solution of the original differential equation.

In [24] it is shown that the solution of Eq. (32) reads:

$$y(x, \kappa, s) = \frac{2s\left(1 - 2\gamma s + \sqrt{1 - 4\gamma s}\right)x\kappa}{\left(2\gamma s x + \kappa\left(1 - 2\gamma s + \sqrt{1 - 4\gamma s}\right)\right)^2} .$$

Then, according to Corollary 8 (part (i)) $y(x, \kappa, s) = \sum_{j=0}^{+\infty} v_j(\kappa, s)x^j$ and the s-representation reads $s = \sum_{j=0}^{+\infty} v_j(\kappa, s)\kappa^j$ (because $Gs = \sum_{j=0}^{+\infty} v_j(\kappa, s)x^j = y$). Now, keeping in mind that $\dfrac{1}{(1-z)^2} = \sum_{j=1}^{+\infty} jz^{j-1}$, one can deduce:

$$y = \frac{2s\left(1 - 2\gamma s + \sqrt{1 - 4\gamma s}\right)x\kappa}{\kappa^2\left(1 - 2\gamma s + \sqrt{1 - 4\gamma s}\right)^2\left(1 - \dfrac{2\gamma s x}{\kappa\left(1 - 2\gamma s + \sqrt{1 - 4\gamma s}\right)}\right)^2} =$$

$$\frac{2sx}{\kappa\left(1 - 2\gamma s + \sqrt{1 - 4\gamma s}\right)}\sum_{j=1}^{+\infty} j\left(\frac{-2\gamma s x}{\kappa\left(1 - 2\gamma s + \sqrt{1 - 4\gamma s}\right)}\right)^{j-1} = 2s\sum_{j=0}^{+\infty}\frac{j(-2\gamma s)^{j-1}}{\left(\kappa\left(1 - 2\gamma s + \sqrt{1 - 4\gamma s}\right)\right)^j}x^j .$$

Now, immediately, $s = 2s\sum_{j=0}^{+\infty}\dfrac{j(-2\gamma s)^{j-1}}{\left(\kappa\left(1 - 2\gamma s + \sqrt{1 - 4\gamma s}\right)\right)^j}\kappa^j$. Therefore, $v_j(\kappa, s) = \dfrac{j(-2\gamma s)^{j-1}}{\left(\kappa\left(1 - 2\gamma s + \sqrt{1 - 4\gamma s}\right)\right)^j}$;

$v(\kappa, s) = \dfrac{s}{\kappa\left(1 - 2\gamma s + \sqrt{1 - 4\gamma s}\right)}$ and $\operatorname{Ker} D_y = \left\{\sum_{k=0}^{+\infty} a_k\left(\dfrac{s}{\kappa\left(1 - 2\gamma s + \sqrt{1 - 4\gamma s}\right)}\right)^k \middle| a_k \in \mathbf{R}\right\}$.

6. Concluding remarks

A number of examples are used to illustrate the functionality of the proposed technique. It becomes clear that the identification of invariants can be as much complex problem as the solution of the initial ODE problem. A necessity of an explicit algorithm for the construction of invariants becomes obvious for more demanding and complex nonlinear ODE problems. The construction of this algorithm is the primary objective of the second part of this paper. Theoretical results presented in the first part serve as a foundation for the construction of this algorithm. Moreover, the concept of the generalized multiplicative operator provides the insight into the algebraic structure of solutions to nonlinear ODE which cannot be identified using conventional exponential operators.

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