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Be careful with the Exp-function method – Additional remarks

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ABSTRACT

It is shown that the solution produced by the Exp-function method may not hold for all initial conditions. Riccati and Maccari nonlinear differential equations are used to illustrate that fact. Conditions of existence for the produced solution in the space of initial conditions and in the space of system's parameters are derived using the operator method based on the generalized operator of differentiation. The concept of the expansion of an ordinary differential equation is introduced and it is shown that the algebraic–analytical solution of Maccari equation can be produced by solving Riccati equation.

1. Introduction

One of the well-known techniques used to seek analytical–algebraic solutions of differential equations is the Exp-function method. It has been demonstrated that the Exp-function method, with the help of symbolic computation, provides a powerful mathematical tool for solving high-dimensional nonlinear evolutions in mathematical physics. The Exp-function method has been exploited for the determination of exact solutions of many nonlinear differential equations [1–9]; we give only a few of many references available.

On the other hand, a straightforward application of the Exp-function method has also attracted a considerable amount of criticism [10–13]. Seven typical errors done when using the Exp-function method are discussed and illustrated in [14].

The objective of this paper is to show that the solution produced by the Exp-function method may not hold for all initial conditions. Moreover, the Exp-function method cannot be used to derive conditions of existence for the produced solution – neither in the space of initial conditions, nor in the space of system's parameters. Finally, we will show that the applicability of the Exp-function method can be redundant – sometimes the partial solution can be produced by solving a much simpler problem.

We will use two differential equations to illustrate and ground the above-stated assertions. The first one is the Riccati equation:

$$y'_x = a(y - y_1)(y - y_2); \qquad a, y_1, y_2 \in R. \tag{1}$$

It can be noted that the general form of the Riccati equation comprises a, y_1 and y_2 as functions of x [15], but we will consider only a simplified form when a, y_1 and y_2 are constants (Eq. (1)). By the way, even a simpler form of the transformed

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Riccati equation is solved in [16] employing the Exp-function method (where it is assumed that $y_2 = -y_1 = y_0$; see Eq. (5) in [16]).

The second differential equation is the transformed Maccari equation [17]:

$$w''_{xx} = b\left(w^2 - w_0^2\right)w; \qquad b, w_0 \in R. \tag{2}$$

It is produced from the Maccari's system [18] by using soliton-type substitutes. Eq. (2) is also solved using the Exp-function method in [17].

As mentioned previously, we will show that solutions of Eq. (2) produced by the Exp-function method do not hold for all initial conditions and for all combinations of system's parameters. Moreover, we will show that there is no need to seek a solution of Eq. (2) expressible in a ratio of finite sums of exponential functions if the solution of Eq. (1) is known beforehand.

2. Auxiliary results

2.1. Preliminary notations

As mentioned in the Introduction, the main objectives of this paper will be illustrated using two differential equations. Moreover, we will expand, narrow and transform these differential equations by changing variables. Therefore it is useful to build a preliminary list of notations of functions of according ordinary differential equations:

(i) The initial differential equation (Riccati equation): y.
(ii) The expanded differential equation (Maccari equation): w.
(iii) The narrowed differential equation: $\tilde{y}$.
(iv) Riccati equation after the change of the variable: u.
(v) Maccari equation after the change of the variable: v.

2.2. Structures of analytical solutions

Let two polynomials are defined as follows:

$$P_1(c,s) = \sum_{k,l \in Z_0} a_{kl} c^k s^l; \qquad P_2(c,s,t) = \sum_{k,l,r \in Z_0} b_{klr} c^k s^l t^r; \tag{3}$$

where a_{kl} and b_{klr} are fixed real (or complex) numbers; c, s and t are real (or complex) variables. Then it is possible to construct two ordinary differential equations with initial conditions:

$$y'_x = P_1(x,y); \qquad y = y(x,c,s); \qquad y(c,c,s) = s \tag{4}$$

and

$$w''_{xx} = P_2\left(x,w,w'_x\right); \qquad w = w(x,c,s,t); \qquad w(c,c,s,t) = s; \qquad w'_x(x,c,s,t)|_{x=c} = t. \tag{5}$$

Usual differentiation operations in respect of variables x, c, s and t are denoted by symbols D_x, D_c, D_s and D_t. Then it is possible to construct generalized differential operators D_y and D_w in respect of variables y and w [19]:

$$\begin{aligned} D_y &:= D_c + P_1(c,s)D_s; \\ D_w &:= D_c + tD_s + P_2(c,s,t)D_t. \end{aligned} \tag{6}$$

These generalized differential operators satisfy all usual relationships of differential operators. For example, the Leibnitz formula holds true:

$$D_w^j(Q_1(c,s,t) \cdot Q_2(c,s,t)) = \sum_{n=0}^{j} \binom{j}{n} \left(D_w^n Q_1(c,s,t)\right)\left(D_w^{j-n} Q_2(c,s,t)\right) \tag{7}$$

where D_w^0 is the identity (unitary) operator $\left(D_w^0 Q_1(c,s,t) = Q_1(c,s,t)\right)$; $D_w^2, D_w^3, \ldots$ are ordinary degrees of linear operators; $Q_m(c,s,t)$; $m = 1,2$ are polynomials of variables c, s and t.

Generalized differential operators D_y and D_w can be exploited to construct analytical solutions $y(x,c,s)$ and $w(x,c,s,t)$ of differential equations (4) and (5) [19]:

$$y = \sum_{j=0}^{+\infty} \frac{(x-c)^j}{j!} D_y^j s; \tag{8}$$

$$w = \sum_{j=0}^{+\infty} \frac{(x-c)^j}{j!} D_w^j s, \tag{9}$$

which converge in some nonempty surrounding $|x - c| < \varepsilon$ in the complex plane. Furthermore, functions $y = y(x, c, s)$ and $w = w(x, c, s, t)$ can be extended into the whole complex plane with the exception of possible singular points.

2.3. Structures of analytical–algebraic solutions

It is important for many engineering applications to obtain analytical–algebraic representations of solutions in the form:

$$y = \sum_{r=1}^{m} \mu_r f_r(\rho(x - c)), \tag{10}$$

where m is a finite constant; $m \in N$; $\mu_r \in C$; f_r and ρ are ordinary functions. We will define the H-rank and associated H-eigenvalues for the construction of special analytical–algebraic solutions.

Let $(p_j;\ j \in Z_0)$ is a sequence of numbers or functions. Then the corresponding sequence of Hankel matrixes $(H_1, H_2, \ldots)$ reads:

$$H_1 := [p_0]; \qquad H_2 := \begin{bmatrix} p_0 & p_1 \\ p_1 & p_2 \end{bmatrix}; \qquad H_3 := \begin{bmatrix} p_0 & p_1 & p_2 \\ p_1 & p_2 & p_3 \\ p_2 & p_3 & p_4 \end{bmatrix}; \cdots \tag{11}$$

and the sequence of determinants of these matrixes is denoted as $(d_k;\ k \in N)$, where $d_k = \det H_k$.

Definition 1. A sequence $(p_j;\ j \in Z_0)$ has an H-rank equal to m if $d_m \neq 0$ but $d_{m+n} = 0$ for all $n \in N$. The following notion will be used throughout the manuscript:

$$Hr(p_j;\ j \in Z_0) = m. \tag{12}$$

Let Eq. (12) holds for a sequence $(p_j;\ j \in Z_0)$. Then it is possible to construct a characteristic H-equation [13]:

$$\det \begin{bmatrix} p_0 & p_1 & \cdots & p_m \\ p_1 & p_2 & \cdots & p_{m+1} \\ \vdots & \vdots & \vdots & \vdots \\ p_{m-1} & p_m & \cdots & p_{2m-1} \\ 1 & \rho & \cdots & \rho^m \end{bmatrix} = 0. \tag{13}$$

Definition 2. Roots $\rho_1, \rho_2, \ldots, \rho_m$ of the characteristic H-equation (13) are H-eigenvalues of the sequence $\left(p_j;\ j \in Z_0\right)$.

Theorem 1. *Let the H-rank of a sequence $(p_j;\ j \in Z_0)$ is m. Moreover, let all H-eigenvalues of that sequence are different: $\rho_k \neq \rho_l$ for $k \neq l$, $(k, l = 1, 2, \ldots, m)$. Then, following equalities hold true:*

$$p_j = \sum_{r=1}^{m} \mu_r \rho_r^j; \qquad j = 0, 1, 2, \ldots. \tag{14}$$

The proof of Theorem 1 is given in [20].

It can be noted that coefficients $\mu_1, \mu_2, \ldots, \mu_m$ can be found solving a linear system of algebraic equations which consists from m different equalities of Eq. (14) (H-eigenvalues $\rho_1, \rho_2, \ldots, \rho_m$ must be determined beforehand). The simplest system is produced when the first m equalities of Eq. (14) are selected (for $j = 0, 1, \ldots, m - 1$). But the same results can be produced for $j = k_1, k_2, \ldots, k_m;\ 0 \leqslant k_1 < k_2 < \cdots < k_m$. Moreover, this linear system of algebraic equations has a unique solution. The sequence defined by Eq. (14) is called an algebraic progression.

Let us assume that the following relationship holds for a sequence $\left(D_y^j s;\ j \in Z_0\right)$:

$$Hr\left(\frac{1}{j!} D_y^j s;\ j \in Z_0\right) = m. \tag{15}$$

Then,

$$D_y^j s = j! \sum_{r=1}^{m} \mu_r \rho_r^j, \tag{16}$$

where $\mu_r = \mu_r(c, s);\ \rho_r = \rho_r(c, s);\ \rho_k \neq \rho_l$ for $k \neq l$.

Eq. (16) can be exploited for the construction of the solution of differential equation (4) in the form of a power series:

$$y = \sum_{j=0}^{+\infty} \frac{(x-c)^j}{j!} D_y^j s = \sum_{j=0}^{+\infty} (x-c)^j \sum_{r=1}^{m} \mu_r \rho_r^j = \sum_{r=1}^{m} \mu_r \sum_{j=0}^{+\infty} (\rho_r (x-c))^j, \tag{17}$$

which converges in the surrounding $|x-c| < \min_r |\rho_r|^{-1}$. But the power series (17) can be extended for all values of the variable x with the exception of such values of x where $\rho_r(x-c) = 1$. In other words, the function $y = y(x,c,s)$ takes the following form:

$$y = \sum_{r=1}^{m} \frac{\mu_r(c,s)}{1 - \rho_r(c,s) \cdot (x-c)}. \tag{18}$$

Eq. (18) is the analytical–algebraic solution of differential equation (4).

2.4. Changing the independent variable of a differential equation

The independent variable of a differential equation can be changed in order to produce a more convenient form of the solution.

Let an invertible function is given by the following equalities:

$$x := \varphi(z); \qquad z = \psi(x). \tag{19}$$

Then following expressions hold:

$$\begin{aligned} z'_x &= \psi'_x = \frac{1}{\varphi'_z}; \\ z''_{xx} &= -\frac{1}{(\varphi'_z)^2} \varphi''_{zz} \cdot z'_x = -\frac{\varphi''_{zz}}{(\varphi'_z)^3}; \\ &\cdots. \end{aligned} \tag{20}$$

Also, for every function

$$y = y(x) = y(\varphi(z)) := u(z) = u(\psi(x)) = u \tag{21}$$

following equalities hold:

$$\begin{aligned} y'_x &= u'_z|_{z=\psi(x)} \cdot \psi'_x = \frac{1}{\varphi'_z} \cdot u'_z|_{z=\psi(x)}; \\ y''_{xx} &= u''_{zz}|_{z=\psi(x)} \cdot (\psi'_x)^2 + u'_z|_{z=\psi(x)} \cdot \psi''_{xx} = \left(\frac{1}{(\varphi'_z)^2} \cdot u''_{zz} - \frac{\varphi''_{zz}}{(\varphi'_z)^3} \cdot u'_z \right)\Bigg|_{z=\psi(x)}; \\ &\cdots. \end{aligned} \tag{22}$$

Thus,

$$\begin{aligned} y'_x|_{x=\varphi(z)} &= \frac{1}{\varphi'_z} \cdot u'_z; \\ y''_{xx}|_{x=\varphi(z)} &= \frac{1}{(\varphi'_z)^2} \cdot u''_{zz} - \frac{\varphi''_{zz}}{(\varphi'_z)^3} \cdot u'_z; \\ &\cdots. \end{aligned} \tag{23}$$

Merging Eq. (23) into one equality and using symbol τ to identify the change of the variable in the differential equation yields:

$$\tau(x; y; y'_x; y''_{xx}; \ldots) := \left(x|_{x=\varphi(z)}; y|_{x=\varphi(z)}; y'_x|_{x=\varphi(z)}; y''_{xx}|_{x=\varphi(z)}; \cdots \right); \tag{24}$$

or

$$\tau(x; y; y'_x; y''_{xx}; \ldots) := \left(\varphi(z); u; \frac{1}{\varphi'_z} \cdot u'_z; \frac{1}{(\varphi'_z)^2} \cdot u''_{zz} - \frac{\varphi''_{zz}}{(\varphi'_z)^3} \cdot u'_z; \cdots \right). \tag{25}$$

3. Expanding and narrowing an ordinary differential equation

Given a differential equation

$$y'_x = P_1(x,y); \qquad y = y(x,c,s); \qquad y = y(c,c,s) = s; \tag{26}$$

its generalized differential operator reads [19]:

$$D_y = D_c + P_1(c,s)D_s; \tag{27}$$

and its solution is described by Eq. (8):

$$y = \sum_{j=0}^{+\infty} \frac{(x-c)^j}{j!} D_y^j s. \tag{28}$$

Following transformations hold:

$$\left(y'_x - P_1(x,y)\right)'_x = y''_{xx} - \left(\frac{\partial P_1(x,y)}{\partial x} + \frac{\partial P_1(x,y)}{\partial y} y'_x\right) = y''_{xx} - \left(\frac{\partial P_1(x,y)}{\partial x} + P_1(x,y)\frac{\partial P_1(x,y)}{\partial y}\right) = 0 \tag{29}$$

where $y = y(x,c,s)$ is defined by Eq. (28).

Let

$$\frac{\partial P_1(x,y)}{\partial x} + P_1(x,y)\frac{\partial P_1(x,y)}{\partial y} := P_2(x,y). \tag{30}$$

Then it is possible to construct the following differential equation:

$$w''_{xx} = P_2(x,w), \tag{31}$$

with $w = w(x,c,s,t)$ and initial conditions $w(c,c,s,t) = s$; $w'_x(x,c,s,t)|_{x=c} = t$. Moreover,

$$w(x,c,s,t) = \sum_{j=0}^{+\infty} \frac{(x-c)^j}{j!} D_w s, \tag{32}$$

where

$$D_w = D_c + tD_s + P_2(c,s)D_t. \tag{33}$$

It follows from Eq. (29) that solutions (28) of the differential equation (26) do satisfy also the differential equation (31). Therefore, such a function $t = f(c,s)$ must exist which makes the following equality to hold true:

$$y(x,c,s) = w(x,c,s,f(c,s)). \tag{34}$$

Now,

$$\sum_{j=0}^{+\infty} \frac{(x-c)^j}{j!} D_y^j s = \sum_{j=0}^{+\infty} \frac{(x-c)^j}{j!} D_w^j s|_{t=f(c,s)}, \tag{35}$$

therefore

$$D_y^j s = D_w^j s|_{t=f(c,s)}; \qquad j = 0,1,2,\ldots. \tag{36}$$

The equality $D_y s = D_w s|_{t=f(c,s)}$ produces $P_1(c,s) = t|_{t=f(c,s)} = f(c,s)$. Thus,

$$f(c,s) = P_1(c,s). \tag{37}$$

Next, the equality $D_y^2 s = D_w^2 s|_{t=f(c,s)}$ yields:

$$\frac{\partial f(c,s)}{\partial c} + f(c,s)\frac{\partial f(c,s)}{\partial s} = P_2(c,s). \tag{38}$$

Thus,

$$D_w^2 s|_{t=f(c,s)} = D_y^2 s = P_2(c,s). \tag{39}$$

Therefore, following equalities hold true:

$$D_w^{n+2} s|_{t=f(c,s)} = D_w^n\left(D_w^2 s\right)\Big|_{t=f(c,s)} = D_y^n\left(D_y^2 s\right) \tag{40}$$

for all $n = 0,1,2,\ldots$.

Thus, finally, following equalities:

$$\left(D_c + tD_s + \left(D_w^2 s\right)D_t\right)^n \left(D_w^2 s\right)|_{t=f(c,s)} = (D_c + f(c,s)D_s)^n \left(D_y^2 s\right) \tag{41}$$

hold for all functions $f(c,s)$ because the differential equation (26) can constructed using the equality (37) for any function.

Definition 3. The differential equation (31) is an expanded differential equation of Eq. (26).

Now, let us assume that a differential equation (31) is given. Also, let a function $f(c,s)$ exists and satisfies the property (38). Then, equalities (41) hold. Moreover, it is possible to construct a differential equation

$$\tilde{y}'_x = f(x,\tilde{y}) \tag{42}$$

with $\tilde{y} = \tilde{y}(x,c,s)$ and the initial condition $\tilde{y}(c,c,s) = s$. The generalized operator of differentiation is then

$$D_{\tilde{y}} = D_c + f(c,s)D_s \tag{43}$$

and the analytical solution reads:

$$\tilde{y} = \sum_{j=0}^{+\infty} \frac{(x-c)^j}{j!} D^j_{\tilde{y}} s. \tag{44}$$

But previous derivations yield:

$$w(x,c,s,f(c,s)) = \tilde{y}(x,c,s). \tag{45}$$

Definition 4. The differential equation (42) is a narrowed differential equation of Eq. (31).

Theorem 2. *Equalities* $D^j_{\tilde{y}} s|_{t=f(c,s)} = D^j_y s;\ j = 0,1,2,\ldots$ *hold if and only Eq. (38) holds true.*

The proof follows from the above derivations.

Corollary 1. *The differential equation (26) is a narrowed differential equation of Eq. (31) and the differential equation (31) is an expanded differential equation of Eq. (26) if and only Eq. (30) holds true.*

In other words, a solution found for the differential equation (26) also satisfies the differential equation (31).

4. Solving Riccati differential equation

It is well known [15] that the partial solution of the Riccati differential equation

$$y'_x = a(y-y_1)(y-y_2); \qquad y = y(x,c,s); \qquad a, y_1, y_2 \in R \tag{46}$$

at $y(c,c,s) = s$ reads:

$$y = \frac{y_2(y_1-s)\exp(ay_1(x-c)) - y_1(y_2-s)\exp(ay_2(x-c))}{(y_1-s)\exp(ay_1(x-c)) - (y_2-s)\exp(ay_2(x-c))}. \tag{47}$$

Nevertheless, we will demonstrate the derivation of this solution by the generalized operator method.

The generalized operator of differentiation for Eq. (46) reads:

$$D_y = D_c + a(s-y_1)(s-y_2)D_s; \tag{48}$$

and the analytical solution takes the form as described by Eq. (8). Since we are interested to obtain an analytical–algebraic representation of this solution in a form comprising a ratio with finite sums of exponential functions in the nominator and the denominator, we perform the change of variables in Eq. (46) [13]:

$$\tau(x,y,y'_x,y''_{xx}) = \left(\frac{1}{\lambda}\ln z,\ v,\ \lambda z v'_z,\ \lambda^2\left(z^2 v''_{zz} + z v'_z\right)\right), \tag{49}$$

where $\lambda \neq 0$; $\lambda \in R$ and $z = \exp(\lambda x)$. This change of variables yields the image differential equation [13] of Eq. (46):

$$v'_z = \frac{a}{\lambda z}(v-y_1)(v-y_2); \qquad v = v(z,\hat{c},s); \qquad v(\hat{c},\hat{c},s) = s; \qquad \hat{c} = \exp(\lambda c). \tag{50}$$

Then, the generalized differential operator of Eq. (50) reads:

$$D_v = D_{\hat{c}} + \frac{a}{\lambda \hat{c}}(s-y_1)(s-y_2)D_s; \tag{51}$$

and the analytical solution reads:

$$v(z,\hat{c},s) = \sum_{j=0}^{+\infty} \frac{(z-\hat{c})^j}{j!} D^j_v s. \tag{52}$$

Let us denote:

$$D^j_v s := \hat{p}_j; \qquad j = 0,1,2,\ldots. \tag{53}$$

Then,

$$\begin{aligned}
&\hat{p}_0 = s;\\
&\hat{p}_1 = \frac{a(s-y_1)(s-y_2)}{\lambda\hat{c}};\\
&\hat{p}_2 = \frac{a^2(s-y_1)(s-y_2)}{\lambda^2\hat{c}^2}\left(2s-y_1-y_2-\frac{\lambda}{a}\right);\\
&\ldots.
\end{aligned} \tag{54}$$

Then, the sequence of Hankel determinants [13] reads:

$$\begin{aligned}
&\det H_1 = \left|\frac{\hat{p}_0}{0!}\right| = s;\\
&\det H_2 = \begin{vmatrix} \frac{\hat{p}_0}{0!} & \frac{\hat{p}_1}{1!} \\ \frac{\hat{p}_1}{1!} & \frac{\hat{p}_2}{2!} \end{vmatrix} = \frac{a(s-y_1)(s-y_2)(asy_1+asy_2-s\lambda-2ay_1y_2)}{2\lambda^2\hat{c}^2};\\
&\det H_3 = \begin{vmatrix} \frac{\hat{p}_0}{0!} & \frac{\hat{p}_1}{1!} & \frac{\hat{p}_2}{2!} \\ \frac{\hat{p}_1}{1!} & \frac{\hat{p}_2}{2!} & \frac{\hat{p}_3}{3!} \\ \frac{\hat{p}_2}{2!} & \frac{\hat{p}_3}{3!} & \frac{\hat{p}_4}{4!} \end{vmatrix} = \frac{a^2Qt(s-y_1)^2(s-y_2)^2(\lambda-a(y_1-y_2))}{144\lambda^6\hat{c}^6};
\end{aligned} \tag{55}$$

where

$$Q = \left\{\begin{aligned} &2s\lambda^3-\lambda^2say_1-5\lambda^2say_2+6\lambda^2ay_1y_2-2\lambda sa^2y_1y_2+4\lambda sa^2y_2^2-2\lambda sa^2y_1^2\\ &+6\lambda y_2a^2y_1^2-6\lambda a^2y_1y_2^2-3sa^3y_1^2y_2-sa^3y_2^3+3sa^3y_1y_2^2+sa^3y_1^3\end{aligned}\right\}.$$

Of course, all calculations are performed using symbolic computation techniques.

We make the proposition that

$$Hr\left(\frac{1}{j!}D_v^j s;\ j\in Z_0\right) = 2 \tag{56}$$

(or equivalently $\det H_3 = 0$) for all $s \in R;\ s \neq y_1;\ s \neq y_2$, but at

$$\lambda = a(y_1-y_2). \tag{57}$$

If the equality (57) hold true, then

$$\begin{aligned}
&\hat{p}_1 = \frac{(s-y_1)(s-y_2)}{\hat{c}(y_1-y_2)};\\
&\hat{p}_2 = \frac{2(s-y_1)^2(s-y_2)}{\hat{c}^2(y_1-y_2)^2}.
\end{aligned} \tag{58}$$

Let us assume that

$$\hat{p}_j = \frac{j!(s-y_1)^j(s-y_2)}{\hat{c}^j(y_1-y_2)^j};\qquad j=1,2,\ldots. \tag{59}$$

But then,

$$\begin{aligned}
\hat{p}_{n+1} &= \left(D_{\hat{c}}+\frac{(s-y_1)(s-y_2)}{\hat{c}(y_1-y_2)}D_s\right)\frac{n!(s-y_1)^n(s-y_2)}{\hat{c}^n(y_1-y_2)^n}\\
&= n!\left(-\frac{n(s-y_1)^n(s-y_2)}{\hat{c}^{n+1}(y_1-y_2)^n}+\frac{(s-y_1)(s-y_2)(n(s-y_1)^{n-1}(s-y_2)+(s-y_1)^n)}{\hat{c}^{n+1}(y_1-y_2)^{n+1}}\right)\\
&= \frac{n!(s-y_1)^n(s-y_2)}{\hat{c}^{n+1}(y_1-y_2)^{n+1}}((n+1)s-(n+1)y_1) = \frac{(n+1)!(s-y_1)^{n+1}(s-y_2)}{\hat{c}^{n+1}(y_1-y_2)^{n+1}}.
\end{aligned} \tag{60}$$

what proves the correctness of Eq. (59).

Now it is easy to prove that all determinants $\det H_j = 0;\ j=3,4,\ldots$ (from elementary properties of the determinant) and our proposition (56) holds (when Eq. (57) holds true).

Then, the image differential equation (50) takes the form:

$$v'_z = \frac{1}{(y_1 - y_2)z}(v - y_1)(v - y_2); \qquad v = v(z, \hat{c}, s); \qquad v(\hat{c}, \hat{c}, s) = s; \qquad \hat{c} = \exp(a(y_1 - y_2)c); \tag{61}$$

and the generalized operator of differentiation reads:

$$D_v = D_{\hat{c}} + \frac{1}{(y_1 - y_2)\hat{c}}(s - y_1)(s - y_2)D_s. \tag{62}$$

Then, the characteristic H-equation (Eq. (13)) reads (at $\lambda = a(y_1 - y_2)$):

$$\begin{vmatrix} s & D_v s & \frac{1}{2}D_v^2 s \\ D_v s & \frac{1}{2}D_v^2 s & \frac{1}{6}D_v^3 s \\ 1 & \rho & \rho^2 \end{vmatrix} = 0. \tag{63}$$

Eq. (63) yields two roots:

$$\rho_1 = 0; \qquad \rho_2 = \frac{s - y_1}{(y_1 - y_2)\hat{c}}. \tag{64}$$

Then, $D_v^j s$ can be expressed in the following form [13]:

$$D_v^j s = j!\left(\mu_1 \rho_1^j + \mu_2 \rho_2^j\right); \qquad j = 0, 1, 2, \ldots. \tag{65}$$

Eq. (65) produces a system of linear algebraic equations for the determination of unknowns $\mu_1, \mu_2 \in R$ ($0^0 = 1$; $0^j = 0$; $j = 1, 2, \ldots$):

$$\begin{cases} s = \mu_1 + \mu_2; \\ \frac{(s-y_1)(s-y_2)}{(y_1-y_2)\hat{c}} = \mu_2 \frac{(s-y_1)}{(y_1-y_2)\hat{c}}. \end{cases} \tag{66}$$

Elementary transformations yield:

$$\mu_1 = y_2; \qquad \mu_2 = s - y_2. \tag{67}$$

Thus,

$$v(z, \hat{c}, s) = \sum_{j=0}^{+\infty} \frac{(z - \hat{c})^j}{j!} D_v^j s = \sum_{j=0}^{+\infty} \frac{(z - \hat{c})^j}{j!} j!\left(y_2 0^j + (s - y_2)\left(\frac{s - y_1}{(y_1 - y_2)\hat{c}}\right)^j\right) = \frac{y_1(s - y_2)\hat{c} - y_2(s - y_1)z}{(s - y_2)\hat{c} - (s - y_1)z} \tag{68}$$

and finally, the analytical–algebraic solution of Riccati differential equation reads:

$$y(x, c, s) = \frac{y_1(s - y_2)\exp(a(y_1 - y_2)c) - y_2(s - y_1)\exp(a(y_1 - y_2)x)}{(s - y_2)\exp(a(y_1 - y_2)c) - (s - y_1)\exp(a(y_1 - y_2)x)}; \tag{69}$$

what after elementary transformations results into Eq. (47).

It can be noted that special solutions can be produced from Eq. (69). When $s = y_k;\ k = 1, 2$, the solution reads:

$$y(x, c, y_k) = \lim_{s \to y_k} y(x, c, s) = y_k; \qquad k = 1, 2. \tag{70}$$

When $y_1 = y_2 = y_0$, the solution reads:

$$y(x, c, s) = \lim_{y_1, y_2 \to y_0} \frac{y_2(s - y_1)\exp(ay_1(x - c)) - y_1(s - y_2)\exp(ay_2(x - c))}{(s - y_1)\exp(ay_1(x - c)) - (s - y_2)\exp(ay_2(x - c))} = y_0 + \frac{s - y_0}{1 + a(y_0 - s)(x - c)}. \tag{71}$$

In this case, when $s = y_0, y(x, c, y_0) = y_0$.

5. Expanding Riccati equation to Maccari differential equation

We exploit relationships (29) and (30) and expand Riccati differential equation (46) to the generalized Maccari differential equation:

$$\begin{aligned} &w''_{xx} = a(2y - (y_1 + y_2))y'_x = 2a^2(w - y_1)(w - y_2)\left(w - \frac{y_1 + y_2}{2}\right); \\ &w = w(x, c, s, t); w(c, c, s, t) = s; w'_x(x, c, s, t)|_{x=c} = t. \end{aligned} \tag{72}$$

It can be noted that the classical Maccari equation (2) is produced when $y_2 = -y_1 = w_0$ (then $b = 2a^2$). Then, the change of variables (50) and the assumption of the condition (57) yield the image Maccari differential equation:

$$u''_{zz} = \frac{1}{(y_1 - y_2)^2 z^2}\Big(2(u - y_1)(u - y_2)\Big(u - \frac{y_1 + y_2}{2}\Big) - zu'_z\Big). \tag{73}$$

The generalized operator of differentiation of the differential equation (73) reads:

$$D_u = D_{\hat{c}} + tD_s + \frac{1}{(y_1 - y_2)^2 \hat{c}^2}\Big(2(s - y_1)(s - y_2)\Big(s - \frac{y_1 + y_2}{2}\Big) - \hat{c}t\Big)D_t. \tag{74}$$

Previous derivations yield:

$$D_u^j s\Big|_{t=\frac{(s-y_1)(s-y_2)}{(y_1-y_2)\hat{c}}} = \Big(D_{\hat{c}} + \frac{(s - y_1)(s - y_2)}{(y_1 - y_2)\hat{c}} D_s\Big)^j s = \frac{j!(s - y_1)^j (s - y_2)}{(y_1 - y_2)^j \hat{c}^j}; \qquad j = 1, 2, \ldots; \; D_v^0 s = D_u^0 s = s. \tag{75}$$

Thus, the solution of the generalized Maccari differential equation reads:

$$u = u\Big(z, \hat{c}, s, \frac{(s - y_1)(s - y_2)}{(y_1 - y_2)\hat{c}}\Big) = v(z, \hat{c}, s) = \frac{y_1(s - y_2)\hat{c} - y_2(s - y_1)z}{(s - y_2)\hat{c} - (s - y_1)z}; \tag{76}$$

and part of solutions of the original Maccari equation take the form described by the equality (69):

$$w(x, c, s, f(c, s)) = y(x, c, s) \tag{77}$$

where

$$f(c, s) = a(s - y_1)(s - y_2). \tag{78}$$

In other words, there is no need to guess the structure of solution of Eq. (2) and to elaborate symbolic computational techniques for the determination of coefficients of that solution as it is done in [17]. It is enough to solve Eq. (1) – what can be done using simple analytical techniques. Of course, the situation is different when the solution of the generalized Maccari equation is to be sought.

6. The existence of the Exp-algebraic solution for the generalized Maccari equation

The generalized Maccari equation can be written in the following form:

$$w''_{xx} = 2a^2(w^3 + \alpha w^2 + \beta w + \gamma); \qquad w = w(x, c, s, t); \qquad w(c, c, s, t) = s; \qquad w'_x(x, c, s, t)|_{x=c} = t, \tag{79}$$

where $\alpha, \beta, \lambda \in R$.

Then the solution of the differential equation (80) can be expressed in the form (69) if and only

$$2a^2(w^3 + \alpha w^2 + \beta w + \gamma) = 2a^2(w - w_1)(w - w_2)\Big(w - \frac{w_1 + w_2}{2}\Big), \tag{80}$$

where $w_1, w_2 \in C$. Thus, according to Viete theorem:

$$\begin{aligned} &\alpha = -\Big(w_1 + w_2 + \frac{w_1 + w_2}{2}\Big); \qquad \beta = w_1 w_2 + w_1 \frac{w_1 + w_2}{2} + w_2 \frac{w_1 + w_2}{2} = w_1 w_2 + \frac{(w_1 + w_2)^2}{2}; \\ &\gamma = -w_1 w_2 \frac{w_1 + w_2}{2}. \end{aligned} \tag{81}$$

Therefore, the equality (80) holds if and only

$$\begin{cases} -\frac{3}{2}(w_1 + w_2) = \alpha, \\ w_1 w_2 + \frac{1}{2}(w_1 + w_2)^2 = \beta, \\ -\frac{1}{2} w_1 w_2 (w_1 + w_2) = \gamma. \end{cases} \tag{82}$$

It is easy to find from the first two equalities of Eq. (82) that $w_1 + w_2 = -\frac{2}{3}\alpha$ and $w_1 w_2 = \frac{9\beta - 2\alpha^2}{9}$. Thus, finally, the third equality of Eq. (82) yields:

$$\gamma = \frac{9\alpha\beta - 2\alpha^3}{27}. \tag{83}$$

It can be noted that terms $(w_1 + w_2)$ and $w_1 w_2$ are expressed through α and β. Then, according to Viete theorem, w_1 and w_2 are the roots of the following algebraic equation:

$$z^2 + \frac{2}{3}\alpha z + \frac{9\beta - 2\alpha^2}{9} = 0. \tag{84}$$

Thus,

$$w_1 = -\frac{\alpha}{3} + \sqrt{\frac{\alpha^2 - 3\beta}{3}}; \qquad w_2 = -\frac{\alpha}{3} - \sqrt{\frac{\alpha^2 - 3\beta}{3}}. \tag{85}$$

Therefore, the generalized Maccari differential equation

$$w''_{xx} = 2a^2\left(w^3 + \alpha w^2 + \beta w + \frac{\alpha(9\beta - 2\alpha^2)}{27}\right) \tag{86}$$

is an expansion of the Riccati differential equation

$$y'_x = a\left(y + \frac{\alpha}{3} - \sqrt{\frac{\alpha^2 - 3\beta}{3}}\right)\left(y + \frac{\alpha}{3} + \sqrt{\frac{\alpha^2 - 3\beta}{3}}\right). \tag{87}$$

Theorem 3. *The solution of the differential equation (80) can be expressed in the form*

$$w = \frac{A_1 \exp(a\lambda_1 x) + A_2 \exp(a\lambda_2 x)}{B_1 \exp(a\lambda_1 x) + B_2 \exp(a\lambda_2 x)} \tag{88}$$

if and only the following relationships hold:

(i) $9\alpha\beta - 2\alpha^3 - 27\gamma = 0;$ (89)

(ii) $t = a\left(s + \frac{\alpha}{3} - \sqrt{\frac{\alpha^2 - 3\beta}{3}}\right)\left(s + \frac{\alpha}{3} + \sqrt{\frac{\alpha^2 - 3\beta}{3}}\right) = a\left(s^2 + \frac{2}{3}\alpha s + \beta - \frac{2\alpha^2}{9}\right).$ (90)

The proof follows from the previous derivations.

We argue that conditions (89) and (89) cannot be derived using the Exp-function method.

7. Computational experiments

Let us consider equation

$$w''_{xx} = 2w^3 - 9w^2 + 13w - 6; \qquad w = w(x, c, s, t); \qquad w(c, c, s, t) = s; \qquad w'_x(x, c, s, t)|_{x=c} = t. \tag{91}$$

It can be noted that Eq. (91) is a generalized Maccari equation because the relationship (89) holds:

$$2w^3 - 9w^2 + 13w - 6 = 2(w-1)(w-2)\left(w - \frac{3}{2}\right). \tag{92}$$

In other words, $y_1 = 1$; $y_2 = 2$ and $a = 1$. We will assume that $c = 0$ in further computations.

Thus, the solution the generalized Maccari equation reads:

$$w = \frac{2(1-s)\exp(x) - (2-s)\exp(2x)}{(1-s)\exp(x) - (2-s)\exp(2x)}. \tag{93}$$

We have shown that Eq. (93) will be the solution of the generalized Maccari equation if and only Eq. (90) holds. Since the roots now are 1 and 2, the condition (90) reads:

$$(s-1)(s-2) - t = 0. \tag{94}$$

We plot the surface $g(s,t) = (s-1)(s-2) - t$ in Fig. 1 (numerical values of $g(s,t)$ higher than 0.5 and lower than -0.5 are truncated to 0.5 and -0.5 accordingly in order to make the figure more clear).

We will check the validity of the produced results by a computational experiment. We will solve the initial problem (91) using approximate computational constant step marching techniques. Lets denote the approximate partial solution $\tilde{y}_k(0 + hk)$; $k = 0, 1, 2, \ldots$; $\tilde{y}_0 = s$, where h is the step size. The analytical–algebraic partial solution (93) is defined on Eq. (94). But we release the constraint (94) and assume that the solution (93) is valid in all space of initial conditions. We travel 100 steps from the predefined initial conditions and compute differences between the approximate computational solution and the analytical "solution" defined by Eq. (93) (Fig. 2). Adding absolute differences for 100 steps produces an error estimate:

$$\varepsilon(s,t) = \sum_{k=1}^{100} |\varepsilon_k| = \sum_{k=1}^{100} \left|\tilde{y}_k(hk) - \frac{2(1-s)\exp(hk) - (2-s)\exp(2hk)}{(1-s)\exp(hk) - (2-s)\exp(2hk)}\right|. \tag{95}$$

The distribution of $\varepsilon(s,t)$ is illustrated in Fig. 3. Numerical values of $\varepsilon(s,t)$ higher than 40 are truncated to 40 in order to make the figure more comprehensive. It can be clearly seen that errors are almost equal to zero on the curve defined by Eq. (94).

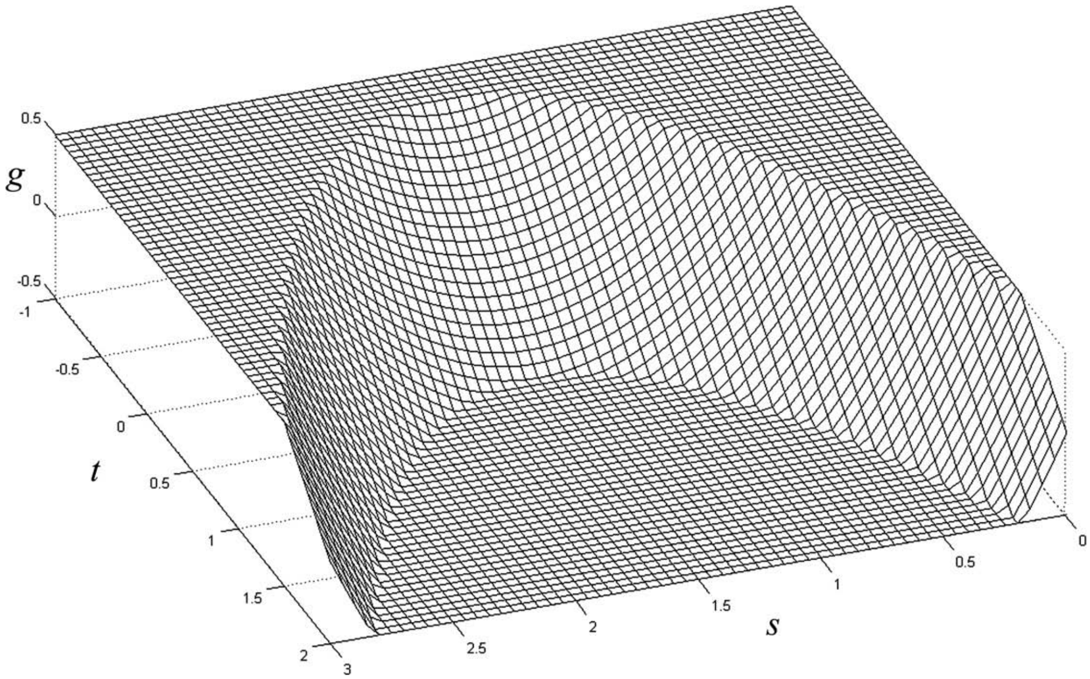

Fig. 1. A graphical representation of the surface $g(s,t) = (s-1)(s-2) - t$.

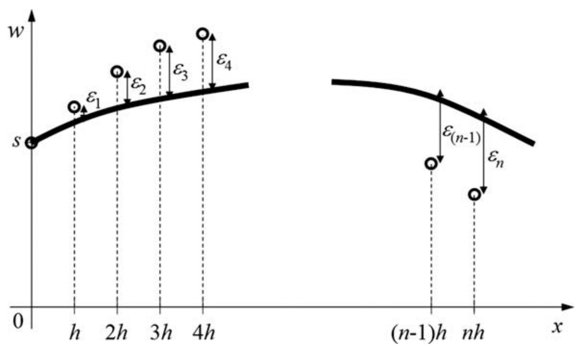

Fig. 2. A schematic diagram illustrating differences between the analytical solution $w(x)$ (the thick solid line) and approximate computational solution (represented by circles) at first n steps after initial conditions $w(0) = s;\ w'_x(0) = t$.

8. Concluding remarks

The Exp-function method can be used for finding exact solutions of nonlinear differential equations with the help of symbolic computation. Unfortunately, there have been a number of cases when straightforward and formal application of the Exp-function method has produced irrelevant results. Seven typical errors done when using the Exp-function method are discussed and illustrated in detail in [14].

This paper demonstrates the eighth and the ninth errors which are frequently done when using the Exp-function method. The eighth error describes the fact that a solution produced by the Exp-function method may not hold for all initial conditions. We argue that the analytical condition describing the existence of the produced solution in the space of initial conditions (or even in the space of the system's parameters) cannot be derived by the Exp-function method. The Exp-function method is based on two main steps (omitting transformations and variable changes leading to a nonlinear ordinary differential equation). The first one is supposing the structure of the algebraic–analytical solution. The second is using symbolic computations for the determination of unknown parameters of the solution. The question about the existence of that solution is typically omitted. The operator method, on the contrary, brings the load of symbolic computations before the structure of the solution is identified. Moreover, the structure of the algebraic-analytic solution is generated automatically together with all conditions of the solution's existence.

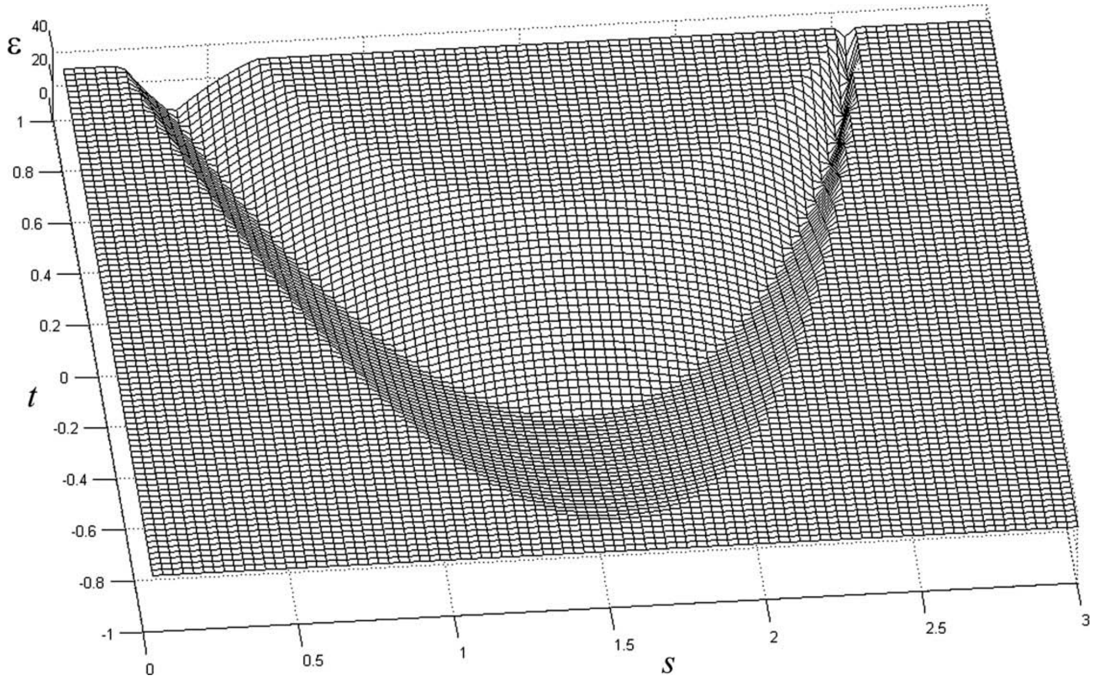

Fig. 3. The distribution of errors between the analytical solution and the computational solution in the parameter plane of initial conditions.

The ninth error is more subtle. It appears that sometimes the Exp-function method is used to solve nonlinear ordinary differential equations which can be produced by the expansion of much simpler differential equations. Clearly, there is no need to elaborate the Exp-function method for expanded differential equations – partial analytical-algebraic solutions satisfy both differential equations.

It can be noted that a number of alternative methods for finding exact solutions of nonlinear differential equations exist. It has been demonstrated that the truncated expansion method [21], the tanh-function method [22], the simplest equation method [23] can be successfully and efficiently used for solving high-dimensional nonlinear differential equations in mathematical physics. The object of this paper is to highlight some problems inherent to the Exp-function method only; all other methods are out of scope of interest in this paper.

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