Computers and Mathematics with Applications 69 (2015) 798–803

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Comments on "The exp-function method and generalized solitary solutions"☆

Z. Navickas [a], T. Telksnys [b], M. Ragulskis [b,*]

[a] Kaunas University of Technology, Department of Applied Mathematics, Studentu 50-325c, LT-51368 Kaunas, Lithuania

[b] Kaunas University of Technology, Research Group for Mathematical and Numerical Analysis of Dynamical Systems, Studentu 50-147, LT-51368 Kaunas, Lithuania

ARTICLE INFO

Article history:
Received 26 June 2014
Accepted 15 February 2015
Available online 6 March 2015

Keywords:
Generalized operator of differentiation
Initial conditions
Exp-function method

ABSTRACT

The author of the commented paper applies the Exp-function method for the derivation of generalized solitary solutions to a system of nonlinear partial differential equations. We argue that the derived solution does not exist for all initial conditions and use operator techniques to derive the conditions of existence of the solitary solution.

1. Introduction

Chang analyzes the following system of nonlinear partial differential equations (PDEs) [1]:

$$\begin{aligned} &u''_{xt} + av'_x v'_t = 0; \\ &v'_t + v'''_{xxx} + (v'_x)^3 + bv'_x u'_{xx} = 0. \end{aligned} \tag{1}$$

Introducing the independent variable substitution $\eta = kx + \omega t$ and denoting $u''_{\eta\eta} = -a(v'_\eta)^2;\ w = v'_\eta$ yield the following nonlinear ordinary differential equation:

$$\omega w + k^3 w''_{\eta\eta} + k^3(1 - ab)w^3 = 0, \tag{2}$$

which can be expressed in the explicit form

$$w''_{\eta\eta} = (ab - 1)w^3 - \frac{\omega}{k^3} w. \tag{3}$$

Chang uses the Exp-function method to obtain solutions in the form of ratios of sums of exponential functions to (3) [1]:

$$w(\eta) = \frac{a_1 \exp(\eta) + a_0 + a_{-1}\exp(-\eta)}{\exp(\eta) + b_0 + b_{-1}\exp(-\eta)}. \tag{4}$$

The aim of this paper is to demonstrate that (4) does not satisfy (3) for all initial conditions. Note that (4) is a solitary solution to (1) and (3) is the Duffing equation without damping.

☆ The paper has been evaluated according to old Aims and Scope of the journal.

* Corresponding author.
E-mail addresses: zenonas.navickas@ktu.lt (Z. Navickas), tadas.telksnys@gmail.com (T. Telksnys), minvydas.ragulskis@gmail.com (M. Ragulskis).

http://dx.doi.org/10.1016/j.camwa.2015.02.017
0898-1221/

2. Solitary solutions to the Duffing equation without damping

2.1. The narrowed differential equation

Consider the first order ordinary differential equation

$$y'_\eta = \alpha y^2 + \beta; \qquad y = y(\eta, c, s); \qquad y(c, c, s) = s, \tag{5}$$

where $\alpha, \beta \in \mathbb{R} \setminus \{0\}$ and s is the initial condition for the solution to (5) formulated at $\eta = c$. The generalized differential operator for (5) reads [2]:

$$D_y = D_c + (\alpha s^2 + \beta) D_s, \tag{6}$$

where D_c, D_s are partial differentiation operators with respect to the index parameters. The solution $y(\eta, c, s)$ to (5) can be expressed in the following form [3,4]:

$$y(\eta, c, s) = \sum_{j=0}^{+\infty} \frac{(\eta - c)^j}{j!} D_y^j s. \tag{7}$$

Eq. (5) is a Riccati equation; its solution is given by [5]:

$$y(\eta, c, s) = \frac{y_2(s - y_1)\exp(\alpha y_1(\eta - c)) - y_1(s - y_2)\exp(\alpha y_2(\eta - c))}{(s - y_1)\exp(\alpha y_1(\eta - c)) - (s - y_2)\exp(\alpha y_2(\eta - c))}, \tag{8}$$

where $y_{1,2} = \pm\sqrt{-\frac{\beta}{\alpha}}$ are the roots of $\alpha y^2 + \beta = 0$.

2.2. The extended differential equation

Differentiating (5) with respect to η yields the following differential equation:

$$y''_{\eta\eta} = 2\alpha^2 y^3 + 2\alpha\beta y, \tag{9}$$

with $y = y(\eta, c, s, t)$ and initial conditions $y(c, c, s, t) = s$; $y'_\eta(\eta, c, s, t)\Big|_{\eta=c} = t$. Denoting y as w and equating the right-hand side coefficients of (3) and (9) produces a system of algebraic equations:

$$\begin{cases} 2\alpha^2 = (ab - 1); \\ 2\alpha\beta = -\dfrac{\omega}{k^3}, \end{cases} \tag{10}$$

that yields:

$$\alpha_* = \pm\sqrt{\frac{ab - 1}{2}}; \qquad \beta_* = \mp\frac{\omega}{\sqrt{2(ab - 1)}k^3}, \tag{11}$$

if only $(ab - 1) > 0$. Notice that $\alpha_*\beta_* < 0$ if $\omega k^3 > 0$. Substituting (11) and $\omega = 2k^3$ into (8) yields:

$$y(\eta, c, s) = -\frac{1}{\alpha_*}\frac{\exp(\eta) - d(c, s)\exp(-\eta)}{\exp(\eta) + d(c, s)\exp(-\eta)}, \tag{12}$$

where $d(c, s) = \exp(2c)\frac{1 + |\alpha_*|s}{1 - |\alpha_*|s}$. Eq. (12) is identical to cases 2 and 3 explicitly formulated in [1]:

$$w(\eta) = \frac{\pm\sqrt{\frac{2}{ab-1}}\exp(\eta) + a_{-1}\exp(-\eta)}{\exp(\eta) \mp a_{-1}\sqrt{\frac{ab-1}{2}}\exp(-\eta)}; \quad \omega = 2k^3, \tag{13}$$

where Chang specifies that a_{-1} is a "free parameter which can be determined according to initial/boundary conditions" [1].

2.3. The generalized differential operator for the extended equation

The solution to (3) is

$$w = w(\eta, c, s, t); \qquad w(c, c, s, t) = s; \qquad w'_\eta(\eta, c, s, t)\Big|_{\eta=c} = t, \tag{14}$$

where s, t are the initial conditions for the solution to (3) formulated at $\eta = c$. The generalized differential operator of (3) with initial conditions (14) reads:

$$D_w = D_c + tD_s + \left((ab - 1)s^3 - \frac{\omega}{k^3}s\right)D_t, \tag{15}$$

where D_c, D_s, D_t are partial differentiation operators with respect to index parameters. Then the solution to (3) reads:

$$w(\eta, c, s, t) = \sum_{j=0}^{+\infty} \frac{(\eta - c)^j}{j!} D_w^j s. \tag{16}$$

2.4. The relationship between the narrowed and the extended equations

The following constraint on the initial conditions s and t is obtained from the structure of the generalized differential operators (15) and (6) for the extended (3) and narrowed (5) differential equations [6]:

$$D_w^j s \bigg|_{t=\alpha_* s^2 + \beta_*} = D_y^j s. \tag{17}$$

This means that Chang's solution (13) satisfies (3) if and only if initial conditions lie on the parabolas $t = \alpha_* s^2 + \beta_*$.

2.5. The interpretation of the constraint

We will use the following notations $A := ab - 1 > 0$ and $B := \frac{\omega}{k^3} > 0$. Eq. (3) can be rewritten as a system of first order ordinary differential equations:

$$\begin{aligned} w'_\eta &= z; \\ z'_\eta &= Aw^3 - Bw. \end{aligned} \tag{18}$$

The system (18) has equilibrium points at $(w_1, z_1) = (0, 0)$ (this equilibrium point is a center) and $(w_{2,3}, z_{2,3}) = \left(\pm\sqrt{\frac{B}{A}}, 0\right)$ (these points are saddles). The first integral of (18) reads:

$$z^2 = \frac{A}{2} w^4 - Bw^2 + 2C, \tag{19}$$

where C is a constant that determines the trajectory in the phase plane (w, z). Setting $w = \pm\sqrt{\frac{B}{A}}$, $z = 0$ and computing C produces the separatrix of (18):

$$z = \pm\sqrt{\frac{A}{2}} \left(w^2 - \frac{B}{A}\right). \tag{20}$$

Note that substituting the values $A = ab - 1$ and $B = \frac{\omega}{k^3}$ into (20) yields

$$z = \pm\sqrt{\frac{ab-1}{2}} w^2 \mp \frac{\omega}{\sqrt{2(ab-1)k^3}}, \tag{21}$$

therefore (21) and (17) define the same curve in the phase plane and the plane of initial conditions respectively. This leads to an important conclusion that the solitary solution to the Duffing equation without damping exists if and only if the initial conditions lie on the separatrix—all other initial conditions yield either periodic finite or infinite solutions (Figs. 1 and 2).

We could finish the discussion here, but will explicitly demonstrate that Chang's solution (13) exists only on the separatrix.

3. Computational experiments

Let $a = b = 2$ and $\omega = 2k^3$. Then the coefficients for the narrowed equation are:

$$\alpha_* = \pm\sqrt{\frac{3}{2}}, \qquad \beta_* = \mp\sqrt{\frac{2}{3}}. \tag{22}$$

The constraints (17) read: $t = \pm\sqrt{6}\left(\frac{s^2}{2} \mp \frac{1}{3}\right)$ (Fig. 3).

Numerical integration can be used to demonstrate that (13) is not a solution to (3) if the constraint $t = \sqrt{6}\left(\frac{s^2}{2} - \frac{1}{3}\right)$ does not hold (Fig. 4). The constraints are further illustrated in Fig. 5. The error Δ between Chang's proposed solution (13) and the actual solution to (3) is estimated using a constant step time-forward integrator (we have used a standard Runge–Kutta fourth order method):

$$\Delta(c, s, t) := \min_{k=1,2} \sum_{j=0}^{100} |w_k(c + jh, c, s, t) - w(c + jh, c, s, t)|, \tag{23}$$

where w_1, w_2 are the functions defined by (13) with positive and negative leading signs in the numerator respectively and w is the solution to (3).

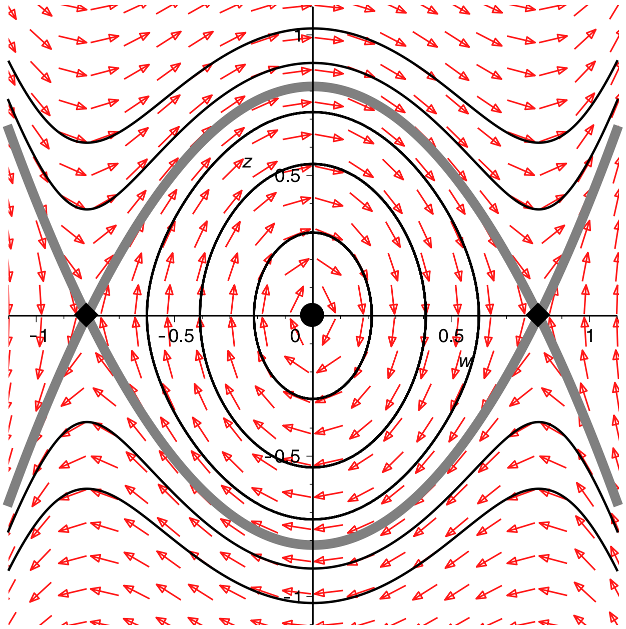

Fig. 1. The phase plane of (18) for $a = b = 2, \omega = 2k^3$. Diamonds denote the saddle points $\left(\pm\sqrt{\frac{2}{3}}, 0\right)$, the black circle denotes the center $(0, 0)$. The thick gray line is the separatrix $z = \pm\sqrt{\frac{3}{2}}w^2 \mp \sqrt{\frac{2}{3}}$. Thin black lines denote periodic finite and infinite solutions.

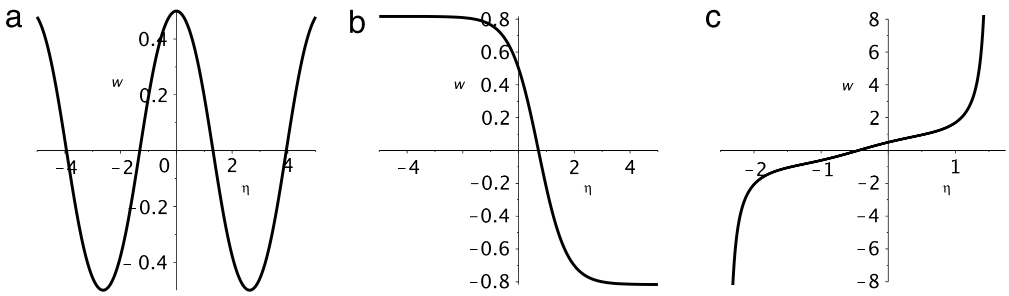

Fig. 2. The evolution of solutions to (3) in respect to η with $a = b = 2$, $\omega = 2k^3$. Initial conditions in (a) are: $c = 0, s = \frac{1}{2}, t = 0$—the solution is bounded and periodic. In (b) initial conditions $c = 0, s = \frac{1}{2}, t = -\frac{5\sqrt{6}}{24}$ yield the solitary solution (12). In (c) initial conditions $c = 0, s = \frac{1}{2}, t = 1$ yield an unbounded solution.

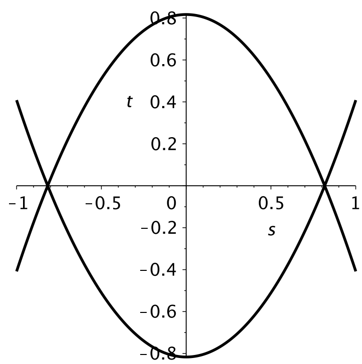

Fig. 3. The constraints on the initial conditions of (3) with $a = b = 2$ and $\omega = 2k^3$ are $t = \pm\sqrt{6}\left(\frac{s^2}{2} \mp \frac{1}{3}\right)$. Note that they coincide with separatrix in Fig. 1.

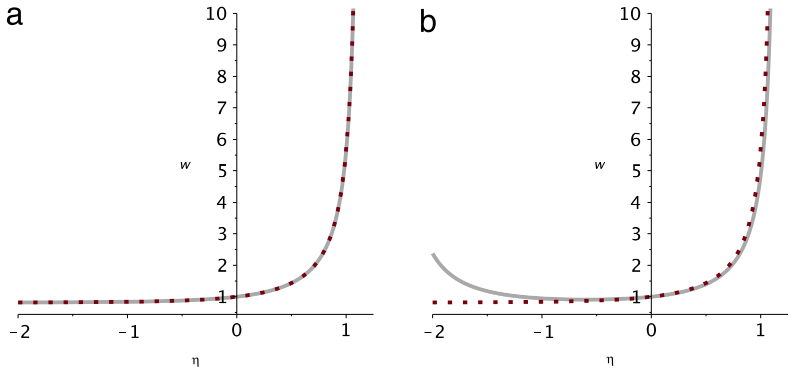

Fig. 4. The solution to (3) with $a = b = 2$ and $\omega = 2k^3$ is the solid gray line; Chang's solution (13) is illustrated by the dotted black line. In (a) the initial conditions are $c = 0$, $s = 1$, $t = \frac{1}{\sqrt{6}}$ (constraint (17) is satisfied). In (b) the initial conditions are $c = 0$, $s = 1$, $t = \frac{1}{\sqrt{6}} - 0.05$ (the constraint (17) does not hold).

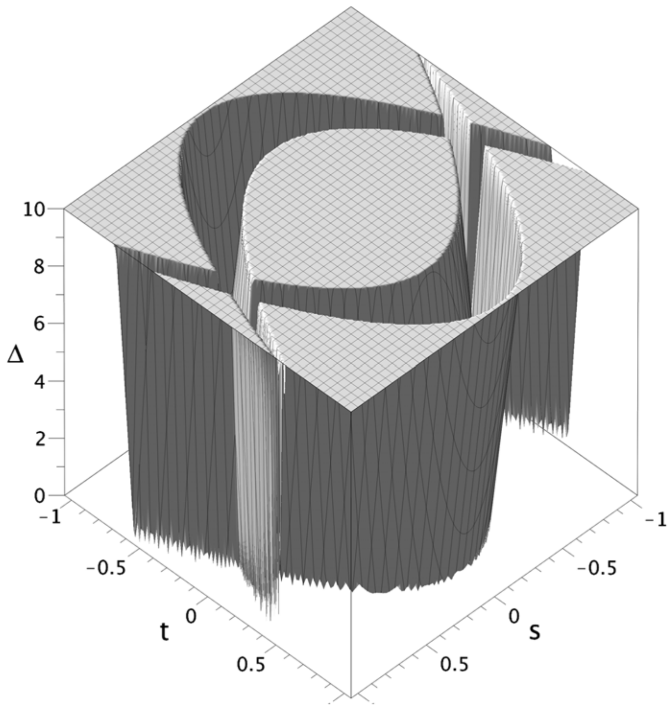

Fig. 5. The graph of the error $\Delta(0, s, t)$ for $s \in [-1, 1]$; $t \in [-1, 1]$ with $a = b = 2$ and $\omega = 2k^3$. Observe that errors are almost equal to zero on the two parabolas $t = \pm\sqrt{6}\left(\frac{s^2}{2} \mp \frac{1}{3}\right)$. Errors over 10 are truncated to 10 for clarity.

4. Concluding remarks

The Exp-function method for the construction of solutions to nonlinear differential equations has attracted a huge amount of criticism during the last years. Seven typical errors done when using the Exp-function method are shown in [7]. Two additional typical errors done when using the Exp-function method are demonstrated in [3]. It is shown in [8] that many solitary solutions to nonlinear differential equations produced by the Exp-function method do not satisfy the original differential equation.

Chang uses the Exp-function method to derive solitary solutions to a nonlinear ordinary differential equation [1]. However, these solutions do not satisfy the differential equation for all initial conditions. We argue that it is not possible to derive these conditions using the Exp-function method and demonstrate the computation scheme that can be used to derive the conditions of existence for the solitary solutions derived in [1].

Acknowledgment

Financial support from the R&D and Innovation Fund of Kaunas University of Technology is acknowledged (Grant number MTEPI-L-14008).

References

[1] J.-R. Chang, The exp-function method and generalized solitary solutions, Comput. Math. Appl. 61 (2011) 2081–2084.
[2] Z. Navickas, L. Bikulciene, Expressions of solutions of ordinary differential equations by standard functions, Math. Model. Anal. 11 (2006) 399–412.
[3] Z. Navickas, M. Ragulskis, How far one can go with the exp function method? Appl. Math. Comput. 211 (2009) 522–530.
[4] Z. Navickas, L. Bikulciene, M. Rahula, M. Ragulskis, Algebraic operator method for the construction of solitary solutions to nonlinear differential equations, Commun. Nonlinear Sci. Numer. Simul. 18 (2013) 1374–1389.
[5] Z. Navickas, L. Bikulciene, M. Ragulskis, Generalization of exp-function and other standard function methods, Appl. Math. Comput. 216 (2010) 2380–2393.
[6] Z. Navickas, M. Ragulskis, L. Bikulciene, Be careful with the exp-function method—additional remarks, Commun. Nonlinear Sci. Numer. Simul. 15 (2010) 3874–3886.
[7] N. Kudryashov, Seven common errors in finding exact solutions of nonlinear differential equations, Commun. Nonlinear Sci. Numer. Simul. 14 (2009) 3507–3529.
[8] P. Ryabov, S. Chesnokov, A note on the ‘‘exp-function method for traveling waves of nonlinear evolution equations’’, Appl. Math. Comput. 218 (2012) 9024–9026.