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ORIGINAL ARTICLE

Solitary solutions to a relativistic two-body problem

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Abstract Necessary and sufficient conditions for the existence of solitary solutions to a generalized model of a two-body problem perturbed by small post-Newtonian relativistic term are derived in this paper. It is demonstrated that kink, bright and dark solitary solutions exist in the model, when the relativistic effects are treated as higher order perturbations. Numerical experiments are used to verify theoretical results.

Keywords Methods: analytical · Waves · Gravitation · Relativistic processes

1 Introduction and motivation

The general relativistic Binet's orbit equation is obtained from the geodesic equation in the Schwarzschild spacetimes (Saca 2008; D'Eliseo 2007, 2009):

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2} + 3\alpha u^2, \tag{1}$$

where $\alpha = \frac{GM}{c^2} \equiv \frac{\mu}{c^2}$ is the gravitational radius of the central body, and c is the speed of light. It is shown in Navickas and Ragulskis (2013) that exact solitary solutions to (1) (with and without discontinuities) do exist in the space of system parameters and initial conditions.

It is shown in Abouelmagd et al. (2015) that the dynamics of a two-body problem perturbed by a small first order post-Newtonian relativistic term can be described by:

$$\frac{d^2u}{d\theta^2} + u = \frac{1}{h^2} + \varepsilon\left(a_0 + a_1 u + u^2 + \left(\frac{du}{d\theta}\right)^2\right), \tag{2}$$

where $a_0 = \frac{2E}{h^2}$ and $a_1 = \frac{4}{h^2}$. A multiple scales method based on the separation of timescales is developed in Abouelmagd et al. (2015) and is applied for the construction of approximate analytical solutions to (2).

It is clear that (2) is a more general than the well-known Binet's equation (1). Thus, immediately the following question arises—do solitary solutions to (2) exist? The main objective of this paper is to give answer to this question.

2 Preliminaries

Solitary waves (also called solitons) are an important phenomenon in mathematical physics. First discovered as solutions to the Korteweg–de-Vries (KdV) equation (Korteweg and de Vries 1985), solitary waves are now considered one of the most vibrant areas of research in both experimental and theoretical physics (Scott 2004; Remoissenet 1999). Solitary waves are often considered in the fields of fluid mechanics (Dauxois and Peyrard 2006; Mercier et al. 2012), optics (Kivshar and Agraval 2003; Chen et al. 2003; Bhrawy et al. 2014), atomic physics (Anglin 2008; Billam and Weiss 2014; Donadello et al. 2014) as well as biophysics and population dynamics (Akhmediev and Ankiewicz 2010; Vitanov and Dimitrova 2010; Vitanov et al. 2009).

Solitary solutions play a fundamental role in optics, which has led to extensive studies of solitary solutions in this

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field. Optical soliton solutions to the Schrödinger equation with quadratic law nonlinearity have been obtained using five integration techniques in Krishnan et al. (2015). A study on the nonlinear dynamics of optical solitons in a cascaded system with Kerr nonlinearity can be found in Zhou et al. (2015). It has been shown that the vector-coupled nonlinear Schrödinger equation admits Thirring combo-soliton solutions in Zhou et al. (2016). Bright, dark and singular solitary solutions in nonlinear directional couplers with a wide class of nonlinearities are analyzed in Savescu et al. (2014).

In recent decades, solitary wave solutions have been discovered in other fields of physics, including astrophysics. Six sets of analytical 2-soliton solutions to the Einstein field equations are constructed in To et al. (1991). The spontaneous solitary space structures have been observed in the Earth's magnetopause by the Cluster spacecraft (Trines et al. 2007). The results of Trines et al. (2007) are extended in Bingham et al. (2008) using a numerical simulation that describes drift wave—zonal flow turbulence to understand self organized solitary wave structures at the magnetopause. The internal structure of a static and spherically symmetric neutron star in the framework of an in-medium modified chiral soliton model in studied in Yakhshiev (2015). Large amplitude slow electron-acoustic solitons in three-electron temperature space plasmas are studied in Mbuli et al. (2015).

3 Do solitary solutions to (2) exist?

Equation (2) can be rewritten into a standardized form as:

$$\frac{d^2u}{d\theta^2}+a\left(\frac{du}{d\theta}\right)^2=b_0+b_1u+b_2u^2, \tag{3}$$

where coefficients a, b_0, b_1, b_2 are as follows: $b_0=\frac{1}{h^2}+\varepsilon a_0$, $b_1=\varepsilon a_1-1$, $b_2=\varepsilon$ and $a=\varepsilon$.

We will consider solitary solutions to have the following form (Navickas et al. 2013; Scott 2004):

$$u(\theta)=\sigma\frac{\prod_{k=1}^{n}(\exp(\eta(\theta-\theta_0))-y_k)}{\prod_{l=1}^{n}(\exp(\eta(\theta-\theta_0))-x_l)}, \tag{4}$$

where $n\in\mathbb{N}$, $\theta_0,\eta,\sigma\in\mathbb{R}$, $y_k,x_l\in\mathbb{C}$. The following derivations demonstrate that (3) does not admit solitary solutions.

3.1 Independent variable transformation

Throughout the paper, the following independent variable transformation will be used:

$$\widehat{\theta}:=\exp(\eta\theta);\qquad \theta=\frac{1}{\eta}\ln\widehat{\theta}. \tag{5}$$

The parameter θ_0 of the solitary solution (4) is then transformed as $\widehat{\theta_0}=\exp(\eta\theta_0)$. For the sake of convenience, the following functions are defined:

$$X(\theta):=\prod_{l=1}^{n}(\theta-x_l);\qquad Y(\theta)=\prod_{k=1}^{n}(\theta-y_k). \tag{6}$$

Using the transformation (5) and (6), the solitary solution (4) reads:

$$\widehat{u}(\widehat{\theta}):=u\left(\frac{1}{\eta}\ln\widehat{\theta}\right)=\sigma\frac{Y(\frac{\widehat{\theta}}{\widehat{\theta_0}})}{X(\frac{\widehat{\theta}}{\widehat{\theta_0}})}. \tag{7}$$

The first and second derivatives of y are transformed as follows:

$$\frac{du}{d\theta}=\eta\widehat{\theta}\frac{d\widehat{u}}{d\widehat{\theta}},\qquad \frac{d^2u}{d\theta^2}=\eta^2\left(\widehat{\theta}\frac{d\widehat{u}}{d\widehat{\theta}}+\widehat{\theta}^2\frac{d^2\widehat{u}}{d\widehat{\theta}^2}\right), \tag{8}$$

thus (3) reads:

$$\eta^2\widehat{\theta}^2\frac{d^2\widehat{u}}{d\widehat{\theta}^2}+\eta^2\widehat{\theta}\frac{d\widehat{u}}{d\widehat{\theta}}+a\eta^2\widehat{\theta}^2\left(\frac{d\widehat{u}}{d\widehat{\theta}}\right)^2=b_0+b_1\widehat{u}+b_2\widehat{u}^2. \tag{9}$$

3.2 Proof that (3) does not admit solitary solutions

Inserting (7) into (9) results in:

$$\frac{Q_4(\widehat{\theta})}{X^4(\frac{\widehat{\theta}}{\widehat{\theta_0}})}+\frac{Q_3(\widehat{\theta})}{X^3(\frac{\widehat{\theta}}{\widehat{\theta_0}})}+\frac{Q_2(\widehat{\theta})}{X^2(\frac{\widehat{\theta}}{\widehat{\theta_0}})}+\frac{Q_1(\widehat{\theta})}{X(\frac{\widehat{\theta}}{\widehat{\theta_0}})}$$
$$=b_0+b_1\frac{Y(\frac{\widehat{\theta}}{\widehat{\theta_0}})}{X(\frac{\widehat{\theta}}{\widehat{\theta_0}})}+b_2\frac{Y^2(\frac{\widehat{\theta}}{\widehat{\theta_0}})}{X^2(\frac{\widehat{\theta}}{\widehat{\theta_0}})}, \tag{10}$$

where the degree of the polynomials Q_4 and Q_3 in $\widehat{\theta}$ is $3n$, while the degree of Q_2 is $2n$ and the degree of Q_1 is n.

As has been shown in Kudryashov and Loguinova (2009), Aslan and Marinakis (2011), Navickas et al. (2016), (9) (and thus (3)) does not admit solitary solutions, because the highest derivative and nonlinear terms are not of balanced degrees in (10).

4 Additional considerations

It is worth noting that the scalar product $\underline{r}\cdot\underline{\dot{r}}$ is approximated by

$$\underline{r}\cdot\underline{\dot{r}}=-h\big(e-e^2\cos f+e^3\cos^2 f+O(e^4)\big), \tag{11}$$

in the process of derivation of (2) (Abouelmagd et al. 2015). Thus, assuming higher orders of e in (11) would result into a more complex form of (2). Without losing the generality, (3) can be rewritten as:

$$\frac{d^2u}{d\theta^2}+a\left(\frac{du}{d\theta}\right)^2=b_0+b_1u+b_2u^2+b_3u^3; \tag{12}$$

$$\frac{d^2u}{d\theta^2}+a\left(\frac{du}{d\theta}\right)^2=b_0+b_1u+b_2u^2+b_3u^3+b_4u^4, \tag{13}$$

what corresponds to higher order approximations of relativistic effects.

Initial conditions for (12) and (13) are formulated as:

$$u(\theta_0)=u_0, \qquad \left.\frac{du}{d\theta}\right|_{\theta=\theta_0}=u_0'. \tag{14}$$

Similar derivations to those in Sect. 3.2 show that (12) does not admit solitary solutions, because the highest derivative and nonlinear terms do not balance. However, it can be demonstrated that (13) does admit both kink (first order, when $n=1$ in (4)) and bright or dark (second order, when $n=2$) solitary solutions.

4.1 Determination of the order of the solitary solution

The first step in determining the existence conditions for (4) in (13) is the determination of the order n of the solitary solution. After transformation of (13), inserting (7) into (13) yields:

$$\frac{P_4(\widehat{\theta})}{X^4(\frac{\widehat{\theta}}{\widehat{\theta}_0})}+\frac{P_3(\widehat{\theta})}{X^3(\frac{\widehat{\theta}}{\widehat{\theta}_0})}+\frac{P_2(\widehat{\theta})}{X^2(\frac{\widehat{\theta}}{\widehat{\theta}_0})}+\frac{P_1(\widehat{\theta})}{X(\frac{\widehat{\theta}}{\widehat{\theta}_0})}$$
$$=b_0+b_1\frac{Y(\frac{\widehat{\theta}}{\widehat{\theta}_0})}{X(\frac{\widehat{\theta}}{\widehat{\theta}_0})}+\cdots+b_4\frac{Y^4(\frac{\widehat{\theta}}{\widehat{\theta}_0})}{X^4(\frac{\widehat{\theta}}{\widehat{\theta}_0})}, \tag{15}$$

where P_4 and P_3 are $3n$th degree polynomials in $\widehat{\theta}$, P_2 is a $2n$th degree polynomial in $\widehat{\theta}$ and P_1 is an nth degree polynomial in $\widehat{\theta}$.

Multiplying both sides of (15) results in:

$$P_4(\widehat{\theta})+X\left(\frac{\widehat{\theta}}{\widehat{\theta}_0}\right)P_3(\widehat{\theta})+X^2\left(\frac{\widehat{\theta}}{\widehat{\theta}_0}\right)P_2(\widehat{\theta})+X^3\left(\frac{\widehat{\theta}}{\widehat{\theta}_0}\right)P_1(\widehat{\theta})$$
$$=b_0X^4\left(\frac{\widehat{\theta}}{\widehat{\theta}_0}\right)+b_1X^3\left(\frac{\widehat{\theta}}{\widehat{\theta}_0}\right)Y\left(\frac{\widehat{\theta}}{\widehat{\theta}_0}\right)$$
$$+b_2X^2\left(\frac{\widehat{\theta}}{\widehat{\theta}_0}\right)Y^2\left(\frac{\widehat{\theta}}{\widehat{\theta}_0}\right)$$
$$+b_3X\left(\frac{\widehat{\theta}}{\widehat{\theta}_0}\right)Y^3\left(\frac{\widehat{\theta}}{\widehat{\theta}_0}\right)+b_4Y^4\left(\frac{\widehat{\theta}}{\widehat{\theta}_0}\right). \tag{16}$$

Both sides of (16) possess a polynomial in $\widehat{\theta}$ of degree $4n$, thus the order of the equation is such that it could admit a solitary solution.

To determine the possible values of n the method described in detail in Navickas et al. (2016) is used. The number of parameters of the differential equation (13) is $u=6$, while the number of parameters in solitary solution is $v=2n+2$, thus the total number of unknown parameters is $p=2n+8$, while the number of linear equations that can be formed by equating the coefficients of (16) is $q=4n+1$. By Navickas et al. (2016), the solitary solution could exist only if the following inequality holds:

$$q\le p, \quad \text{or} \quad 4n+1\le 2n+8, \tag{17}$$

thus the only possible orders for the solitary solution in (13) are $n=1,2,3$.

4.2 Kink solitary solution

The case $n=1$ yields the kink solitary solution, which has the form:

$$u=\sigma\frac{\exp(\eta(\theta-\theta_0))-y_1}{\exp(\eta(\theta-\theta_0))-x_1}. \tag{18}$$

The following subsection is dedicated to the derivation of kink solitary solutions to (13).

4.2.1 The Riccati equation

Suppose the following differential equation is given:

$$\frac{dw}{d\theta}=c_0+c_1w+c_2w^2, \quad w(\theta_0)=w_0, \tag{19}$$

where $c_0, c_1, c_2\in\mathbb{R}$. It is known that all solutions that satisfy the Riccati equation (19) are kink solitary solutions that have the following form (Polyanin and Zaitsev 2003; Kudryashov and Loguinova 2009; Navickas and Ragulskis 2013; Navickas et al. 2013):

$$w=w_2\frac{\exp(c_2(w_1-w_2)(\theta-\theta_0))-\frac{w_1(w_0-w_2)}{w_2(w_0-w_1)}}{\exp(c_2(w_1-w_2)(\theta-\theta_0))-\frac{w_0-w_2}{w_0-w_1}}, \tag{20}$$

where w_1, w_2 are the roots of $c_0+c_1w+c_2w^2$.

4.2.2 Extension of the Riccati equation

Differentiating both sides of (19) by θ yields:

$$\frac{d^2w}{d\theta^2}=(c_1+2c_2w)\frac{dw}{d\theta}$$
$$=c_0c_1+\left(2c_0c_2+c_1^2\right)w+3c_1c_2w^2+2c_2^2w^3. \tag{21}$$

On the other hand, squaring both sides of (19) results in:

$$\left(\frac{dw}{d\theta}\right)^2=c_0^2+2c_1c_2w+\left(2c_0c_2+c_1^2\right)w^2$$

$$+2c_1c_2w^3+c_2^2w^4. \tag{22}$$

Equations (21) and (22) yield:

$$\frac{d^2w}{d\theta^2}+a\left(\frac{dw}{d\theta}\right)^2=c_0c_1+ac_0^2+\left(2ac_0c_1+2c_0c_2+c_1^2\right)w+\left(2ac_0c_2+ac_1^2+3c_1c_2\right)w^2+\left(2ac_1c_2+2c_2^2\right)w^3+ac_2^2w^4. \tag{23}$$

Denoting $w=u$ and renaming the coefficients to b_j, $j=0,\dots,4$ yields (13). Thus, (13) can be obtained by extending the Riccati equation if for some $c_0,c_1,c_2\in\mathbb{R}$, the following relations hold true:

$$b_4=ac_2^2,\qquad b_3=2ac_1c_2+2c_2^2$$
$$b_2=2ac_0c_2+ac_1^2+3c_1c_2; \tag{24}$$

$$b_1=2ac_0c_1+2c_0c_2+c_1^2,\qquad b_0=c_0c_1+ac_0^2. \tag{25}$$

4.2.3 *The construction of the kink solution to (13)*

The theorem on extended and narrowed equations that can be found in Navickas et al. (2010, 2013) applied to (19) and (13) results in the following statement.

Corollary 4.1 *Equation* (13) *has kink solitary solutions if conditions* (24), (25) *hold true.*

Proof Suppose the conditions (24), (25) hold true. Then all solutions to (19) also satisfy (13), however, the solutions to (13) are also solutions to (19) if the following constraint on the initial conditions (14) holds true:

$$u_0'=c_0+c_1u_0+c_2u_0^2,\qquad u_0\in\mathbb{R}. \tag{26}$$

□

4.2.4 *Computational experiments*

Suppose the following differential equation of the form (13) is given:

$$\frac{d^2u}{d\theta^2}+2\left(\frac{du}{d\theta}\right)^2=66-35u-19u^2+6u^3+2u^4. \tag{27}$$

Immediately from (27) it follows that $a=2$. Next, conditions (24), (25) must be tested. Solving (24) with respect to c_0,c_1,c_2 yields the values: $c_0=\mp 6$, $c_1=\pm 1$, $c_2=\pm 1$. Inserting these values into (25) yields valid equalities, so it can be concluded that (27) is the extension of the following Riccati equations:

$$\frac{dw}{d\theta}=\pm\left(-6+w+w^2\right). \tag{28}$$

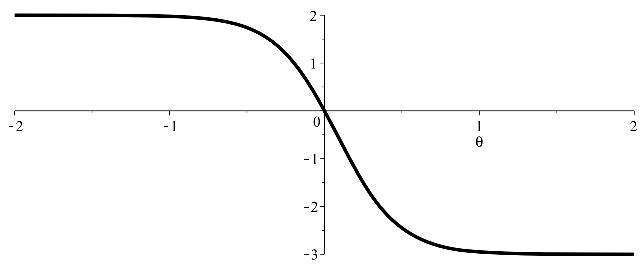


Fig. 1 Graph of the kink solitary solution (29) with $\theta_0=u_0=0$

The roots of $6+w+w^2$ are $w_1=2$, $w_2=-3$, thus by Corollary (4.1), the kink solution to (27) reads:

$$u=-3\frac{\exp(\pm 5(\theta-\theta_0))+\frac{2(u_0+3)}{3(u_0-2)}}{\exp(\pm 5(\theta-\theta_0))-\frac{u_0+3}{u_0-2}}. \tag{29}$$

A graph of (29) is given in Fig. 1. The solution (29) is valid only if the initial conditions u_0,u_0' satisfy

$$u_0'=\pm\left(-6+u_0+u_0^2\right). \tag{30}$$

The necessity of imposing the condition (30) for the solution (29) to hold true can be demonstrated using numerical integration. The notation $\widehat{u}(\theta;\theta_0,u_0,u_0')$ is used for the approximate numerical solution for the initial condition values of θ_0,u_0,u_0'; the notations $\widehat{w}_1(\theta;\theta_0,w_0)$, $\widehat{w}_2(\theta;\theta_0,w_0)$ are used to denote the approximate numerical solutions to (28) with the $+$ and $-$ signs respectively.

A constant step, time-forward numerical integrator RK4 is used to compute the approximations. The error Δ of N forward steps of length h between the approximations is evaluated using the formula:

$$\Delta\left(\theta_0,u_0,u_0'\right)=\min_{j=1,2}\sum_{k=0}^{N}\left|\widehat{u}\left(\theta_0+kh;\theta_0,u_0,u_0'\right)-\widehat{w}_j(\theta_0+kh;\theta_0,u_0)\right|. \tag{31}$$

The plot of Δ is given in Fig. 2. It can be observed that the error between the solutions of (27) and (28) is almost zero on the parabolas defined by (30).

4.3 Bright and dark solitary solutions

The case $n=2$ yields the solitary solution of the form

$$u=\sigma\frac{(\exp(\eta(\theta-\theta_0))-y_1)(\exp(\eta(\theta-\theta_0))-y_2)}{(\exp(\eta(\theta-\theta_0))-x_1)(\exp(\eta(\theta-\theta_0))-x_2)}. \tag{32}$$

The form (32) can represent a variety of solitary solutions, including those with singularities, however, the most physical significance is attributed to bright and dark solitary solutions (Scott 2004; Yang 2010).

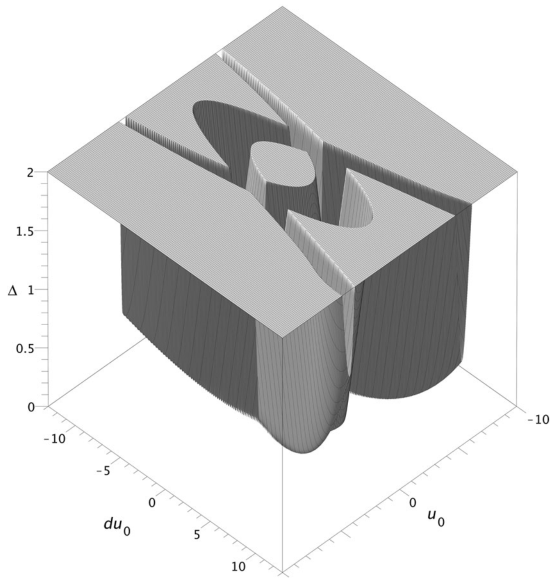


Fig. 2 Graph of the error Δ for $-10 \le u_0 \le 10$ and $-15 \le u_0' \le 15$. Note that the errors are almost equal to zero on the parabolas (30)

4.3.1 *Narrowed equation for bright and dark solitary solutions*

Let us consider the following differential equation:

$$\frac{\mathrm{d}w}{\mathrm{d}\theta} = \gamma(w-\beta)\sqrt{\alpha_0+\alpha_1 w+w^2}; \quad w(\theta_0)=w_0, \tag{33}$$

where $\beta, \alpha_0, \alpha_1, \gamma \in \mathbb{R}$.

It is known that the solutions of (33) are only of the form (32) (Telksnys et al. 2016). The parameters of the solution to (33) are given in Telksnys et al. (2016).

4.3.2 *The extended equation of (33)*

Differentiating both sides of (33) yields:

$$\begin{aligned}\frac{\mathrm{d}^2w}{\mathrm{d}\theta^2} &= \frac{\frac{\mathrm{d}w}{\mathrm{d}\theta}(4w^2+(3\alpha_1-2\beta)w+2\alpha_0-\beta\alpha_1)}{2\sqrt{\alpha_0+\alpha_1 w+w^2}}\\ &= 2\gamma^2w^3+\frac{3}{2}\gamma^2(\alpha_1-2\beta)w^2\\ &\quad+\gamma^2(\beta^2-2\beta\alpha_1+\alpha_0)w+\frac{1}{2}\gamma^2\beta(\beta\alpha_1-2\alpha_0).\end{aligned} \tag{34}$$

Squaring both sides of (33) results in:

$$\begin{aligned}\left(\frac{\mathrm{d}w}{\mathrm{d}\theta}\right)^2 &= \gamma^2w^4+\gamma^2(\alpha_1-2\beta)w^3\\ &\quad+\gamma^2(\beta^2-2\beta\alpha_1+\alpha_0)w^2+\gamma^2\beta(\beta\alpha_1-\alpha_0)w\\ &\quad+\gamma^2\beta^2\alpha_0.\end{aligned} \tag{35}$$

From (34) and (35) it follows that:

$$\begin{aligned}&\frac{\mathrm{d}^2w}{\mathrm{d}\theta^2}+a\left(\frac{\mathrm{d}w}{\mathrm{d}\theta}\right)^2\\ &\quad= a\gamma^2w^4+\gamma^2\big(2+a(\alpha_1-2\beta)\big)w^3\\ &\qquad+\gamma^2\left(\frac{3}{2}\alpha_1-3\beta+a\big(\beta^2-2\beta\alpha_1+\alpha_0\big)\right)w^2\\ &\qquad+\gamma^2\big(a\beta(\beta\alpha_1-2\alpha_0)+\beta^2-2\beta\alpha_1+\alpha_0\big)w\\ &\qquad+\gamma^2\beta\left(\beta\left(a\alpha_0+\frac{1}{2}\alpha_1\right)-\alpha_0\right).\end{aligned} \tag{36}$$

Renaming the function $w = u$ and denoting the coefficients of $w^k, k = 0, \ldots, 4$ as b_k, (13) is obtained. It follows from these derivations that (13) is the extended equation produced from (33) if there exists such $\beta, \alpha_0, \alpha_1, \gamma$ that the coefficients of (13) satisfy the following relations:

$$b_4 = a\gamma^2, \qquad b_3 = \gamma^2\big(2+a(\alpha_1-2\beta)\big); \tag{37}$$

$$b_2 = \gamma^2\left(\frac{3}{2}\alpha_1-3\beta+a\big(\beta^2-2\beta\alpha_1+\alpha_0\big)\right); \tag{38}$$

$$b_1 = \gamma^2\big(a\beta(\beta\alpha_1-2\alpha_0)+\beta^2-2\beta\alpha_1+\alpha_0\big); \tag{39}$$

$$b_0 = \gamma^2\beta\left(\beta\left(a\alpha_0+\frac{1}{2}\alpha_1\right)-\alpha_0\right). \tag{40}$$

4.3.3 *The construction of bright and dark solitary solutions to (13)*

The following statement can be verified using the theorem that can be found in Navickas et al. (2010, 2013).

Corollary 4.2 *Equation* (13) *has bright or dark solitary solutions if conditions* (37)–(40) *hold true.*

Proof Suppose the conditions (37)–(40) hold true. Then all solutions to (33) also satisfy (13), however, solutions to (13) satisfy (33) only if the following condition on the initial conditions of (13) holds true:

$$u_0' = \gamma(u_0-\beta)\sqrt{\alpha_0+\alpha_1u_0+u_0^2}, \quad u_0 \in \mathbb{R}. \tag{41}$$

□

4.3.4 *Computational experiments*

Let us consider the following differential equation:

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}-\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 = -40-128u-110u^2-36u^3-4u^4. \tag{42}$$

Note that $a = -1$ in (42). The conditions (37)–(40) are tested by solving (37)–(39) for $\gamma, \beta, \alpha_1, \alpha_0$. The solutions

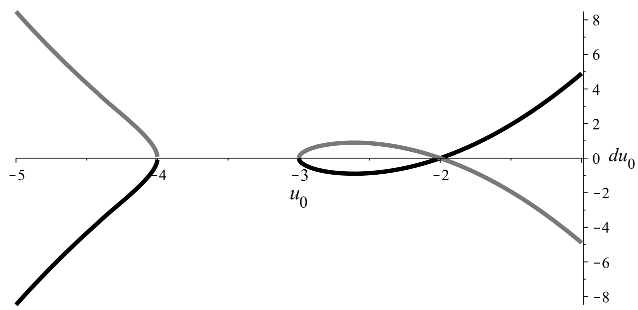


Fig. 3 The *curve* in the phase plane defined by (44). The *black line* denotes the trajectory of the solitary solution to (43) with the + sign, the *gray line* denotes the trajectory of the solitary solution to (43) with the − sign

are $\gamma = \pm 2, \beta = -2, \alpha_1 = 7, \alpha_0 = 12$. Note that there six solutions for $\gamma, \beta, \alpha_1, \alpha_0$ in total, however, only two yield (33) with real coefficients. Inserting the values into (40), equality is obtained.

Thus the differential equation (42) is the extended equation of:

$$\frac{\mathrm{d}w}{\mathrm{d}\theta} = \pm 2(w+2)\sqrt{12+7w+w^2}. \tag{43}$$

Using the solution provided in Telksnys et al. (2016), the solution to (42) is constructed as the solution to (43), when the initial conditions satisfy the following constraint:

$$u_0' = \pm 2(u_0+2)\sqrt{12+7u_0+u_0^2}, \quad u_0 \in \mathbb{R}. \tag{44}$$

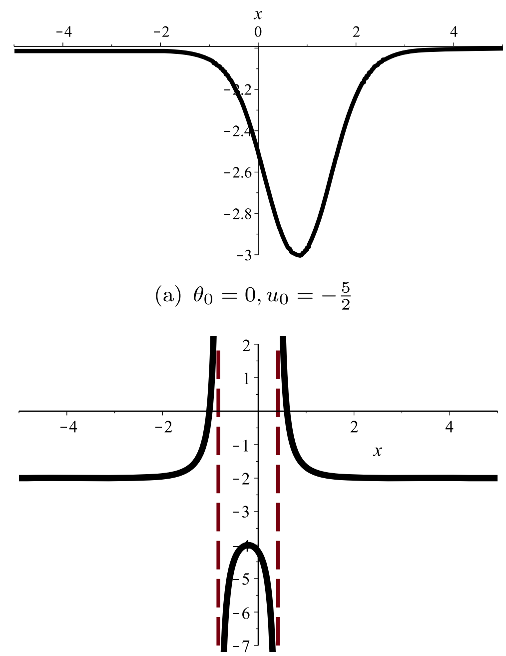
The condition (44) is plotted in Fig. 3. Note that the properties of the solitary solution depend on the selection of the initial condition u_0. Selecting $-3 \le u_0 < -2$ yields the dark solitary solution (Fig. 4a), the range $u_0 \le -4$ and $u_0 > -2$ yield solitary solutions with two singularities (Fig. 4b). Note that the solitary solution does not exist for $-4 < u_0 < -3$.

Performing the same numerical experiments for (42), (43) as in Sect. 4.2.4 yields the error graph displayed in Fig. 5.

5 Concluding remarks

It is demonstrated that solitary solutions exist in the relativistic two-body problem when the relativistic effects are treated as higher order perturbations. Conditions for the existence of the solitary solution are derived in the terms of the equation parameters and the initial conditions. Computational experiments are used to verify the theoretical results.

Solitary wave solutions, emphasized in many fields of physics, are also of definite interest in astrophysics. Our results on (13) could be valuable to the study of the dynamics of the two-body problem perturbed by post-Newtonian relativistic term. Construction of the solitary solutions to other



(a) $\theta_0 = 0, u_0 = -\frac{5}{2}$

(b) $\theta_0 = 0, u_0 = -\frac{21}{5}$

Fig. 4 Solitary solutions to (42) for initial conditions satisfying (44). **a** is the dark solitary solution, **b** is the solitary solution with two singularities. The *dashed line* in (**b**) denotes the singularity points of the solitary solution

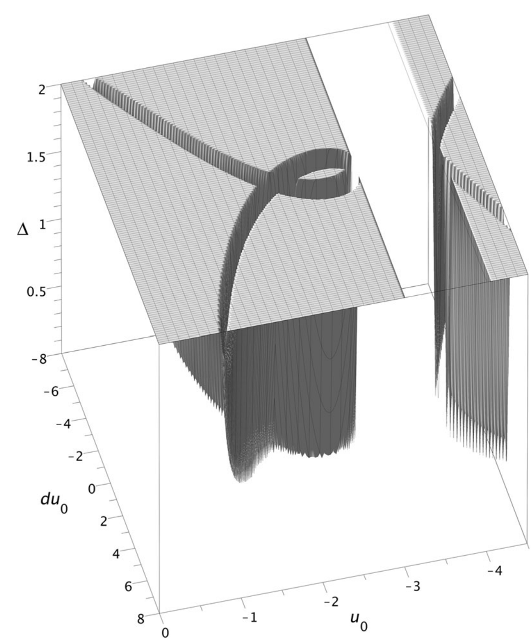


Fig. 5 Graph of the error Δ for $-5 \le u_0 \le 0$ and $-8 \le u_0' \le 8$. Note that the errors are almost equal to zero on *curves* defined by (43) and shown in Fig. 4

models of astrophysics remains a definite objective of future research.

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