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LETTER

Comments on "Two exact solutions to the general relativistic Binet's equation"

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Abstract In their recent manuscript He and Zeng claim that they have solved the general relativistic Binet's orbit equation using the exp-function method and have obtained two exact solutions useful for theoretical analysis. We argue that the obtained solutions do not satisfy the original differential equation. Moreover, we present the alternative framework for the solution of the general relativistic Binet's orbit equation.

Keywords Binet's orbit equation · Exp-function method · Solitary solution

1 Introduction

He and Zeng (2009) search for new exact solutions to the general relativistic Binet's orbit equation, which is obtained from the geodesic equation in the Schwarzschild space-times (Saca 2008; D'Eliseo 2007, 2009):

$$\frac{d^2u}{d\varphi^2} + u = \frac{\mu}{l^2} + 3\alpha u^2, \tag{1}$$

where $\alpha = GM/c^2 \equiv \mu/c^2$ is the gravitational radius of the central body, and c is the speed of light.

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He and Zeng (2009) assume that the solution to (1) can be expressed in the form

$$\hat{u}(\varphi) = \frac{a_1 \exp(\varphi) + a_0 + a_{-1}\exp(-\varphi)}{\exp(\varphi) + b_0 + b_{-1}\exp(-\varphi)}, \tag{2}$$

and determine constants a_1, a_0, a_{-1}, b_0 and b_{-1} by substituting (2) into (1) and by equating the coefficients of $\exp(n\varphi)$ ($n = -3; -2; -1; 0; 1; 2$ and 3) to be zero. By doing that, He and Zeng claim to use the Exp-function method (He and Wu 2006) for solving the Binet's equation and do obtain two sets of constants (He and Zeng 2009):

$$\begin{aligned}
a_0 &= \frac{1}{2B}\left(1 + \sqrt{1-4BA}\right),\\
a_1 &= a_0,\\
a_{-1} &= -\frac{4}{5}a_0^2 B/\left(1 + \sqrt{1-4BA}\right),\\
b_0 &= -a_0\left(-\frac{1}{2} + \frac{1}{2}\sqrt{1-4BA}\right)\Big/A,\\
b_{-1} &= \frac{1}{(-3 + 2\sqrt{1-4BA})}\frac{2a_0^2}{5A^2}\left(\frac{5}{2} - \frac{5}{2}\sqrt{1-4BA}\right.\\
&\qquad \left. - 9BA + 2AB\left(1 + \sqrt{1-4BA}\right)\right),
\end{aligned} \tag{3}$$

and

$$\begin{aligned}
a_0 &= \frac{1}{2B}\left(1 + \sqrt{1-4BA}\right),\\
a_1 &= a_0,\\
a_{-1} &= -\frac{4}{5}a_0^2 B/\left(1 + \sqrt{1-4BA}\right),\\
b_0 &= -a_0\left(-\frac{1}{2} + \frac{1}{2}\sqrt{1-4BA}\right)\Big/A,
\end{aligned} \tag{4}$$

$$b_{-1} = \frac{1}{(-3-2\sqrt{1-4BA})} \frac{2a_0^2}{5A^2} \left(\frac{5}{2} + \frac{5}{2}\sqrt{1-4BA} - 9BA + 2AB\left(1+\sqrt{1-4BA}\right) \right),$$

where

$$A = \frac{\mu}{l^2}, \qquad B = 3\alpha. \tag{5}$$

He and Zeng claim that these are two new exact solutions to (1) and discuss that both solutions can represent some physically realizable situations, which "deserve further study" (He and Zeng 2009).

The main objective of this paper is to show that the supposed solution in the form described by (3) does not satisfy (1).

2 The structure of solutions

A straightforward application of the Exp-function method for the construction of solutions to nonlinear differential equations has attracted a considerable amount of criticism. Seven typical errors done when using the Exp-function method are discussed and illustrated in Kudryashov (2009). Two additional typical errors done when using the Exp-function method are debated in Navickas et al. (2010). It is shown in Ryabov and Chesnokov (2012) that many solitary solutions to nonlinear differential equations produced by the Exp-function method do not satisfy the original differential equation.

An alternative operator based analytical criterion determining if an exact solution of a nonlinear differential equation can be found by the Exp-function method is derived in Navickas and Ragulskis (2009). The employment of this criterion provides the structure of the solution. The load of symbolic calculations is brought before the structure of the solution is identified. This is in contrary to the Exp-function type methods where the structure of the solution is first guessed, and then symbolic calculations are exploited for the identification of unknown parameters.

First of all, let us note that (1) is the KdV type equation in the transformed coordinate system. The KdV equation

$$\frac{\partial w}{\partial t} - 6w\frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^3} = 0 \tag{6}$$

was discovered in 1895 by Korteweg and de Vries (1895). This equation is a paradigmatic partial differential equation in nonlinear physics; it models one-dimensional shallow water waves with small but finite amplitudes. (6) is integrable and the Cauchy problem for this equation can be solved (Gardner et al. 1967; Lax 1968; Hirota 1971). This equation has soliton, rational and elliptic solutions (Airault et al. 1977; Rosales 1978). The change of variables based on the existence of a propagating wave $w(x,t) = u(x-st)$ (where s is the constant wave speed) yields an ordinary differential equation:

$$-(s+6u)\frac{du}{d\varphi} + \frac{d^3u}{d\varphi^3} = 0, \tag{7}$$

where φ is the variable of the propagating wave $\varphi = x - st$. Integration of (7) with respect to φ yields

$$-(s+3u)u + \frac{d^2u}{d\varphi^2} = 3C; \quad C = const. \tag{8}$$

(8) can be rewritten in the form:

$$\frac{d^2u}{d\varphi^2} = \gamma^2(u-c_1)(u-c_2); \quad c_1, c_2, \gamma \in \mathbf{C}, \tag{9}$$

where $c_{1,2} = -\frac{s}{6} \pm \sqrt{\frac{s^2}{36} - C}$; $\gamma^2 = 3$. As mentioned previously, (1) is the KdV type equation in the transformed coordinate system. Really, (1) can be rearranged into the form

$$\frac{d^2u}{d\varphi^2} = B\left(u^2 - \frac{1}{B}u + \frac{A}{B}\right). \tag{10}$$

Denoting roots of the polynomial $u^2 - \frac{1}{B}u + \frac{A}{B}$ by

$$c_{1,2} = \frac{1}{2B}\left(1 \pm \sqrt{1-4AB}\right) \tag{11}$$

helps to simplify (10) to the form

$$\frac{d^2u}{d\varphi^2} = B(u-c_1)(u-c_2). \tag{12}$$

The solution to (12) expressible in a ratio of sums of exponentials or powers of sums of exponentials reads (Navickas et al. 2013)

$$u(\varphi,\varphi_0,u_0) = c_k + \frac{12(c_k-c_l)(u_0-c_k)\exp(\sqrt{B(c_k-c_l)}(\varphi-\varphi_0))}{((\sqrt{3(c_k-c_l)} \pm \sqrt{2u_0-3c_l+c_k}) + (\sqrt{3(c_k-c_l)} \mp \sqrt{2u_0-3c_l+c_k}) \cdot \exp(\sqrt{B(c_k-c_l)}(\varphi-\varphi_0)))^2}$$
$$k,l = 1,2;\ k \neq l, \tag{13}$$

where φ_0 is the point of initial conditions; $u_0 = u(\varphi_0)$. This solution does not satisfy (12) for all initial conditions. It does exist only on the curve in the parameter plane of initial conditions (Navickas et al. 2013):

$$3(\dot{u}_0)^2 = (u_0 - c_k)^2\sqrt{2u_0 - 3c_l + c_k}; \quad k, l = 1, 2;\ k \neq l, \tag{14}$$

where $\dot{u}_0 = \frac{du(\varphi)}{d\varphi}|_{\varphi=\varphi_0}$.

The expression of solution (13) can be rearranged noting that (11) yields equalities

$$c_2 - c_1 = -\frac{1}{B}\sqrt{1-4AB},$$

$$2u_0 - 3c_1 + c_2 = 2u_0 + \frac{1}{B} - \frac{2}{B}\sqrt{1-4AB}.$$

We argue that there does not exist any other form of solutions to (1) expressible in a ratio of sums of exponentials or powers of sums of exponentials (rigorous proofs and derivations are given in Navickas et al. 2013).

3 Computational experiments

Let us consider the following differential equation:

$$\frac{d^2u}{dx^2} = u^2 - u - 2. \tag{15}$$

Note that $u^2 - u - 2 = (u-2)(u+1)$. Therefore $\gamma^2 = 1$; $c_1 = 2$; $c_2 = -1$. Then, $A = -2$; $B = 1$; $\sqrt{1-4AB} = \pm 3$; $c_1 - c_2 = 3$; $2\dot{u}_0 - 3c_2 + c_1 = 2\dot{u}_0 + 5$.

We set $k = 1$ and $l = 2$ in (13) because we consider only real solutions, and assume $\varphi_0 = 0$. Thus, the solution reads:

$$u_1(\varphi, 0, u_0) = 2 + \frac{36(u_0 - 2)\exp(\sqrt{3}\varphi)}{((3+\sqrt{2u_0+5}) + (3-\sqrt{2u_0+5})\exp(\sqrt{3}\varphi))^2};$$

$$u_2(\varphi, 0, u_0) = 2 + \frac{36(u_0 - 2)\exp(\sqrt{3}\varphi)}{((3-\sqrt{2u_0+5}) + (3+\sqrt{2u_0+5})\exp(\sqrt{3}\varphi))^2}. \tag{16}$$

Let $u_0 = 2$. Then $u_1(\varphi, 0, 2) = u_2(\varphi, 0, 2) \equiv 2$ at $\dot{u}_0 = 0$ (the condition for $\dot{u}_0$ follows from (14)).

Now, let us consider the following initial conditions: $\varphi_0 = 0$ and $u_0 = -2$. Then (16) yields two solutions:

$$u_1(\varphi, 0, -2) = 2 - 72\frac{\exp(\sqrt{3}\varphi)}{(2 + \exp(\sqrt{3}\varphi))^2} \tag{17}$$

which does exist at $\dot{u}_0 = -\frac{4}{\sqrt{3}}$; and

$$u_2(\varphi, 0, -2) = 2 - 72\frac{\exp(\sqrt{3}\varphi)}{(1 + 2\exp(\sqrt{3}\varphi))^2} \tag{18}$$

which does exist at $\dot{u}_0 = \frac{4}{\sqrt{3}}$.

Graphs of solutions $u_1(\varphi, 0, -2)$ and $u_2(\varphi, 0, -2)$ are shown in Fig. 1.

Note that (17) and (18) describe the instantaneous shape of a solitary solution (hence the association with the KdV equation). Really,

$$\lim_{\varphi\to\pm\infty} u_1(\varphi, 0, -2) = \lim_{\varphi\to\pm\infty} u_{-2}(\varphi, 0, -2) = 2;$$

graphs of $u_1(\varphi, 0, -2)$ and $u_2(\varphi, 0, -2)$ are shown in Fig. 2; conditions of their existence are shown in Fig. 3. In fact, relationship (14) singles out solitary solutions to the Binet's equations from the whole family of existing solutions. Moreover, the structure of this solitary solution (13) is not

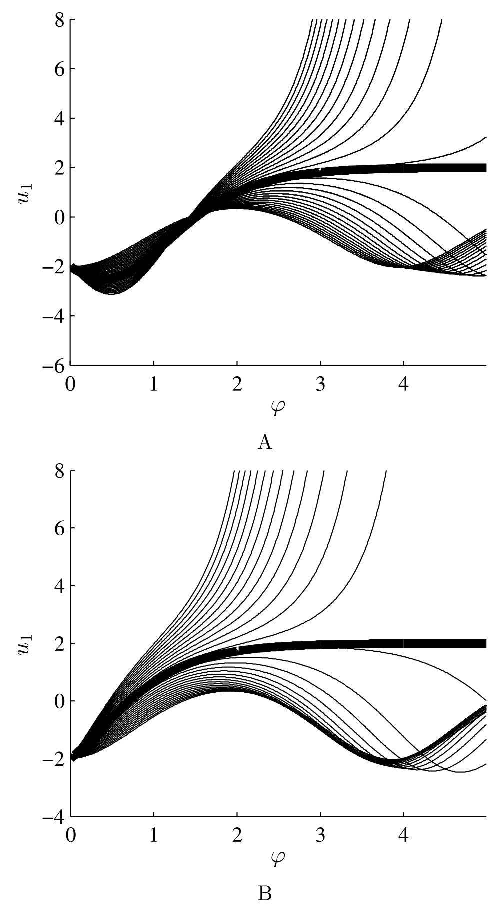

Fig. 1 Solutions to $\ddot{u} = u^2 - u - 2$ at $u_0 = -2$. Solutions at $\dot{u}_0 = -4j/30$; $j = 1, 2, \ldots, 30$ are illustrated in part (**A**); the *thick solid line* represents $u_1(\varphi, 0, -2)$. Solutions at $\dot{u}_0 = 4j/30$; $j = 1, 2, \ldots, 30$ are illustrated in part (**B**); the *thick solid line* represents $u_2(\varphi, 0, -2)$

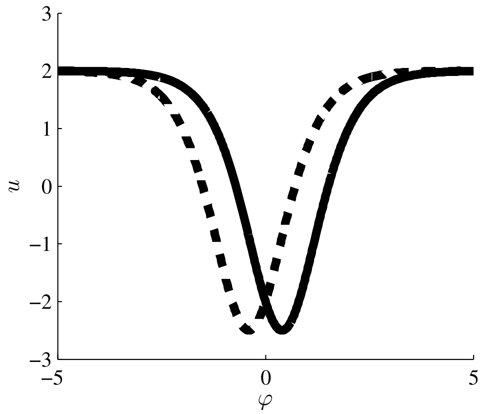

Fig. 2 Solutions to $\ddot{u} = u^2 - u - 2$ at $u_0 = -2$. The *thick solid line* represents $u_1(\varphi, 0, -2)$; the *thick dashed line* represents $u_2(\varphi, 0, -2)$

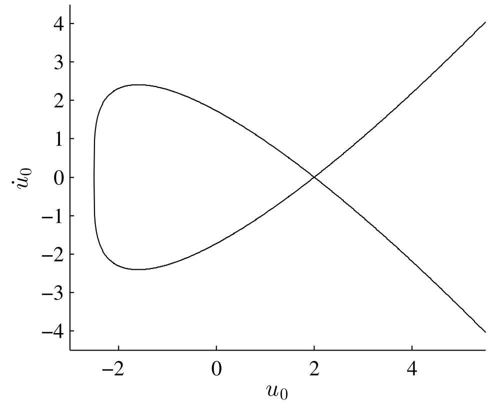

Fig. 3 A graphical interpretation of the constraint $3(\dot{u}_0)^2 = (u_0 - 2)^2\sqrt{2u_0 + 5}$

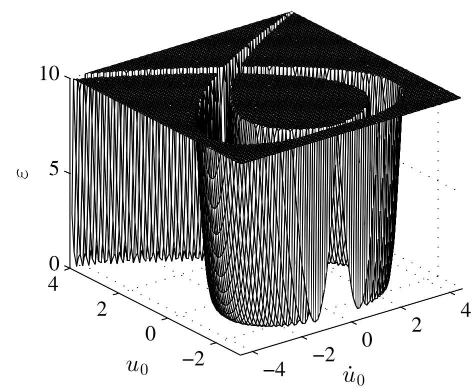

Fig. 4 The distribution of $\varepsilon(u_0, \dot{u}_0)$ for $\ddot{u} = u^2 - u - 2$

supposed, not assumed or guessed at the beginning of analysis (in contrary to He and Zeng 2009), but is generated automatically by the algebraic operator method (Navickas et al. 2013).

We will check the validity of results by performing the following computational experiment. We will construct the approximate numerical solution to (15) at initial conditions $u_0 = u(0)$ and $\dot{u}_0 = \frac{du(\varphi)}{d\varphi}|_{\varphi=0}$ using constant step time forward marching techniques. Let us denote the approximate computational solution as $\tilde{u}(0 + jh)$; $j = 0, 1, 2, \dots,$ where h is the step size. The exact analytical solution (13) is defined on the parameter line (14), but we release the constraint (14) and assume that the solution (13) is valid throughout the plane of initial conditions. We travel 200 steps from the preselected pair of initial conditions and compute differences between the approximate numerical solution and the exact solution (13). Adding absolute differences for 200 steps produces an error estimate:

$$\varepsilon(u_0, \dot{u}_0) = \min_{k=1,2} \sum_{j=1}^{200} \left|\tilde{u}(0 + jh) - u_k(0 + jh, 0, u_0)\right|. \quad (19)$$

The distribution of $\varepsilon(u_0, \dot{u}_0)$ is shown in Fig. 4; numerical values of $\varepsilon(u_0, \dot{u}_0)$ higher than 10 are truncated to 10 in order to make the figure more comprehensive. It is clear that errors are almost equal to zero on the curves defined by (14).

Finally, it can be noted that “solutions” (2) to (15) produced by He and Zang read (the set of parameters (3) yields (20); the set of parameters (4) yields (21)):

$$\hat{u}_1(\varphi) = \frac{10\exp(\varphi) + 10 - 4\exp(-\varphi)}{5\exp(\varphi) - 5 - 2\exp(-\varphi)}, \quad (20)$$

$$\hat{u}_2(\varphi) = -\frac{9}{5} \cdot \frac{-5\exp(\varphi) + 5 - 2\exp(-\varphi)}{18\exp(\varphi) - 9 - 4\exp(-\varphi)}. \quad (21)$$

Note that the denominator of (20) is equal to zero when $\exp(\varphi) = \frac{5+\sqrt{65}}{10}$ and the denominator of (21) is equal to zero when $\exp(\varphi) = \frac{9+\sqrt{369}}{36}$. In other words, the “solution” (20) has an infinite discontinuity at $\varphi \approx 0.2671$ and the “solution” (21) has an infinite discontinuity at $\varphi \approx -0.2440$. Graphs of (20) and (21) are illustrated in Fig. 5 and do not have any meaningful physical interpretation.

He and Zeng (2009) write that the first solution “leads to an unstable orbit” (probably they mean that $\hat{u}_1(\varphi)$ has an infinite discontinuity at $\varphi > 0$), but the second solution “leads to the motion of the perihelion with accuracy better than that given in all GR treatises”. The last claim is completely ungrounded. First of all, function $\hat{u}_2(\varphi)$ does not satisfy the original differential equation (1). In principle, one can assume that $\hat{u}_2(\varphi)$ is a “solution” to (1) at $\varphi > 0$. Note that $\hat{u}_2(0) = -\frac{72}{25}$. Thus, one can set one initial condition $u_0 = -\frac{72}{25}$ and vary the other initial condition $\dot{u}_0$ trying to find an initial solution similar to $\hat{u}_2(\varphi)$. Unfortunately, it is impossible to find such initial conditions which would result into a shape similar to $\hat{u}_2(\varphi)$ (Fig. 6).

Secondly, bounded real solutions to (15) do not exist at $u_0 = -\frac{72}{25}$. Functions $u_1(\varphi, 0, u_0)$ and $u_2(\varphi, 0, u_0)$ are not real at $u_0 = -\frac{72}{25}$ because $2u_0 + 5 = -\frac{19}{25} < 0$ (16). This

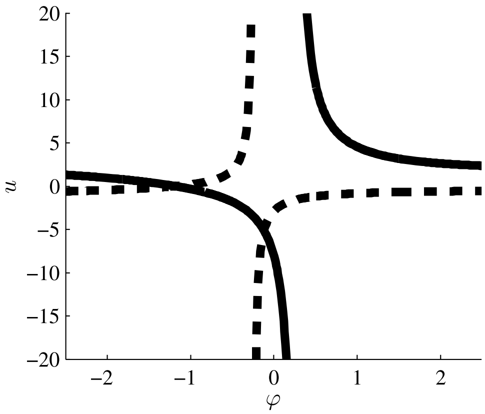

Fig. 5 The *thick solid line* stands for $\hat{u}_1(\varphi)$ and the *thick dashed line* stand for $\hat{u}_2(\varphi)$ computed for the set of parameters representing $\ddot{u} = u^2 - u - 2$

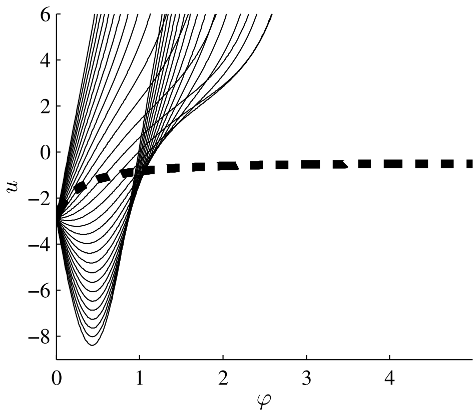

Fig. 6 Solutions to $\ddot{u} = u^2 - u - 2$ at $u_0 = -\frac{72}{25}$. Solutions at $\dot{u}_0 = 20 - \frac{40}{30}j$; $j = 1, 2, \ldots, 30$ are illustrated by *thin solid lines*; the *thick dashed line* represents $\hat{u}_2(\varphi)$

result is not astonishing. One should use proper techniques for finding solutions to nonlinear differential equations.

4 Concluding remarks

The variety of solutions to (1) is much wider than solutions expressed in a ratio of sums of exponentials or powers of sums of exponentials. But if someone wishes to seek such solutions, one needs to use appropriate techniques. The solution method proposed in He and Zeng (2009) is a typical example when the much criticized Exp-function method leads to wrong results.

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