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Generalisations of the compound trapezoidal rule

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Abstract

A set of symmetric equidistant quadrature rules with equal internal weights are developed. Some of the rules use the first (and second) derivative(s) in addition to the function values of the integrand. A single unifying concept based on finite element techniques is used to develop this set of quadrature rules. With focus on the quadratures rules' degree of precision we compute the quadrature weights and error estimates for all rules given.

Keywords: Trapezoidal rule; Finite element method; Nodal derivatives; Nodal weights

1. Introduction

A definite integral $I(f) := \int_a^b f(t)\,dt$ can be approximated by an integration rule $R(f) = \sum_{i=1}^{n} \omega_i f(t_i)$ where t_i are the nodes and ω_i are the weights of the n-point integration rule R. Automatic algorithms are widely used for the numerical approximation of definite integrals. A survey of automatic integration routines is available in [4–8,10,11, 16,17].

In many engineering applications quadrature routines of high degree of precision must be used in real-time mode. Unfortunately, compound Newton–Cotes quadrature formulas [6] require that the number of intervals must be a composite numeral. It means that a significant number of nodes at the end of an experimental data series must be deleted and the integration interval artificially shortened, especially if the number of nodes is not known at the beginning of the experiment. Of course one could combine the compound rules with a fractional rule at the end of the experimental data series. Thus all data-points and the pre-defined degree of precision would be kept, but the overall symmetry would be lost. The object of this paper is to develop such quadrature formulas which would keep the overall symmetry and be applicable in real time applications. One of examples of such formulas is presented in [15].

However, not only discrete nodal values of the integrated function are available in certain occasions. A typical example is an integrator of second-order ordinary differential equations, when data on solutions' magnitude, its first and second derivatives are available at every time node. It is clear that evaluation of these derivatives would increase the accuracy of a definite integral of the solution calculated in the time domain. Such quadrature routines involving Hermite nodal conditions are available in [6].

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Naturally, there exists a definite need for integration rules not only capable of coping with the nodal Hermite conditions, but also being insensitive to the number of intervals. Such class of quadrature routines can be named as the generalisation of the compound trapezoidal rule. Such nomination has a definite sense. The compound trapezoidal rule integrates polynomials of the first degree exactly; the internal weights are equal to one; there are no requirements for the number of nodes except that it must be greater or equal to 2.

In this paper we generalise the trapezoidal rule in the sense of the degree of exactly integrated polynomial and the number of nodal derivatives. The classical trapezoidal rule evaluates only the discrete values of the integrated function. We propose a quadrature routine, which degree of precision and the number of nodal derivatives can be selected freely while the weights of internal nodes are equal. It can be noted that the time steps in the domain of integration are fixed and equal, so partition adaptability is impossible.

2. Illustrative example

Let us denote the values of a function f and its derivatives at time moments $t_0 + (i-1)\cdot h$, $i = 1, \dots, 8$ as:

$$f_i = f\big(t_0 + (i-1)\cdot h\big); \qquad f_i' = f'\big(t_0 + (i-1)\cdot h\big); \qquad f_i'' = f''\big(t_0 + (i-1)\cdot h\big), \tag{1}$$

where h is a time step and superscripts denote the first and the second derivatives.

One-dimensional finite elements consisting from 3 nodes will be used for approximation of the function in the domain of integration. The co-ordinate of the first node of the first finite element E_1 is t_0, the co-ordinates of the second and the third nodes are $t_0 + h$ and $t_0 + 2h$. The second finite element E_2 is shifted by interval h in respect of the first finite element. Thus the co-ordinates of the first, the second and the third nodes of E_2 are $t_0 + h$, $t_0 + 2h$ and $t_0 + 3h$. The co-ordinates of the nodes of the last (the sixth) finite element are $t_0 + 5h$, $t_0 + 6h$ and $t_0 + 7h$ (Fig. 1).

Let us introduce local co-ordinate ζ for every finite element. The co-ordinates of the first, the second and the third nodes are assumed to be -1, 0 and 1 in the local co-ordinate system. Then, the shape functions of a finite element are constructed as eighth degree polynomials satisfying Hermite conditions of interpolation. In general all nine shape functions and their first and second derivatives are zero in -1, 0, 1 with the following nine exceptions (one exception per shape function):

$$N_1(-1) = N_2(0) = N_3(1) = N_4'(-1) = N_5'(0) = N_6'(1) = N_7''(-1) = N_8''(0) = N_9''(1) = 1, \tag{2}$$

where superscripts denote derivatives by ζ. Co-ordinate t and local co-ordinate ζ are related as:

$$t = t_0 + (k + \zeta)\cdot h, \quad k = 1, \dots, 6. \tag{3}$$

Thus,

$$f(\zeta_i) = f_{k+i-1}; \qquad f'(\zeta_i) = h \cdot f_{k+i-1}'; \qquad f''(\zeta_i) = h^2 \cdot f_{k+i-1}''; \quad i = 1, \dots, 3. \tag{4}$$

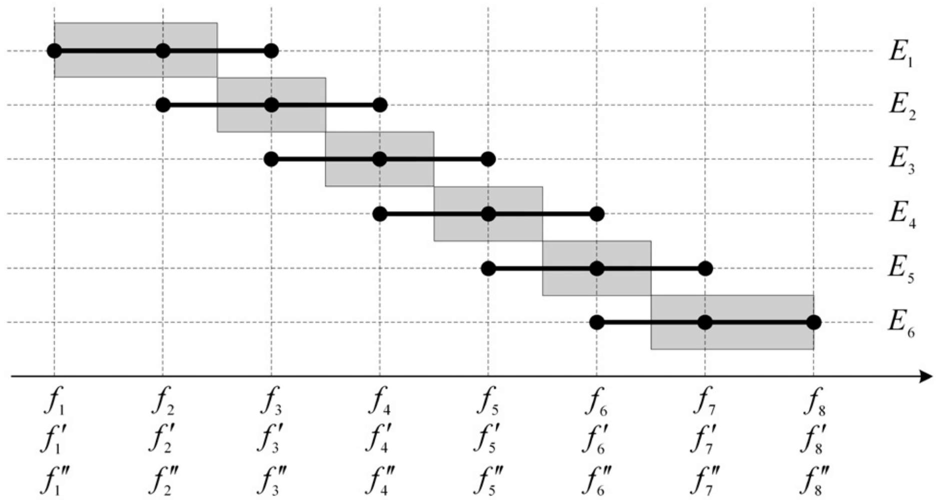

Fig. 1. Direct stiffness procedure at $m = 3$, $n = 8$.

A continuous function can be interpolated in the domain of a finite element through the nodal values of that function (f_k, f'_k and f''_k at node 1; f_{k+1}, f'_{k+1} and f''_{k+1} at node 2; f_{k+2}, f'_{k+2} and f''_{k+2} at node 3):

$$\begin{aligned}\hat{f}(\zeta) &= f_k \cdot N_1(\zeta) + f_{k+1} \cdot N_2(\zeta) + f_{k+2} \cdot N_3(\zeta) \\ &\quad + \left(f'_k \cdot N_4(\zeta) + f'_{k+1} \cdot N_5(\zeta) + f'_{k+2} \cdot N_6(\zeta)\right) \cdot h \\ &\quad + \left(f''_k \cdot N_7(\zeta) + f''_{k+1} \cdot N_8(\zeta) + f''_{k+2} \cdot N_9(\zeta)\right) \cdot h^2. \end{aligned} \tag{5}$$

It is clear that the interpolated function $\hat{f}(\zeta)$ is a polynomial of the eighth degree.

We calculate the integral in the global domain $t_0 \leqslant t \leqslant t_0 + 7h$ in the following way (Fig. 1):

$$\int_{t_0}^{t_0+7h} f(t)\,dt = \int_{t_0}^{t_0+\frac{3}{2}h} f(t)\,dt + \int_{t_0+\frac{3}{2}h}^{t_0+\frac{5}{2}h} f(t)\,dt + \cdots + \int_{t_0+\frac{9}{2}h}^{t_0+\frac{11}{2}h} f(t)\,dt + \int_{t_0+\frac{11}{2}h}^{t_0+7h} f(t)\,dt. \tag{6}$$

Eq. (3) and the assumption that the function f is a polynomial of the eighth degree leads to the following expression of the first integral on the right side of Eq. (6):

$$\int_{t_0}^{t_0+\frac{3}{2}h} f(t)\,dt = h\int_{-1}^{0.5} f(\zeta)\,d\zeta = h\sum_{i=1}^{3}\left(f_i\int_{-1}^{0.5} N_i(\zeta)\,d\zeta + hf'_i\int_{-1}^{0.5} N_{3+i}(\zeta)\,d\zeta + h^2 f''_i\int_{-1}^{0.5} N_{6+i}(\zeta)\,d\zeta\right). \tag{7}$$

Analogously, the second integral on the right side of Eq. (6) takes the form:

$$\begin{aligned}\int_{t_0+\frac{3}{2}h}^{t_0+\frac{5}{2}h} f(t)\,dt &= h\int_{-0.5}^{0.5} f(\zeta)\,d\zeta \\ &= h\sum_{i=1}^{3}\left(f_{i+1}\int_{-0.5}^{0.5} N_i(\zeta)\,d\zeta + hf'_{i+1}\int_{-0.5}^{0.5} N_{3+i}(\zeta)\,d\zeta + h^2 f''_{i+1}\int_{-0.5}^{0.5} N_{6+i}(\zeta)\,d\zeta\right).\end{aligned} \tag{8}$$

Similar procedure can be continued with all middle integrals until the last one ($t = t_0 + (6+\zeta)h$):

$$\begin{aligned}\int_{t_0+\frac{11}{2}h}^{t_0+7h} f(t)\,dt &= h\int_{-0.5}^{1} f(\zeta)\,d\zeta \\ &= h\sum_{i=1}^{3}\left(f_{i+5}\int_{-0.5}^{1} N_i(\zeta)\,d\zeta + hf'_{i+5}\int_{-0.5}^{1} N_{3+i}(\zeta)\,d\zeta + h^2 f''_{i+5}\int_{-0.5}^{1} N_{6+i}(\zeta)\,d\zeta\right).\end{aligned} \tag{9}$$

The definite integrals of the shape functions in Eqs. (7)–(9) can be calculated explicitly:

$$\int_{-1}^{0.5} N_1(\zeta)\,d\zeta = \frac{468\,627}{1\,146\,880}; \quad \int_{-1}^{0.5} N_4(\zeta)\,d\zeta = \frac{72\,567}{1\,146\,880}; \quad \int_{-1}^{0.5} N_7(\zeta)\,d\zeta = \frac{4329}{1\,146\,880};$$

$$\int_{-1}^{0.5} N_2(\zeta)\,d\zeta = \frac{19\,359}{17\,920}; \quad \int_{-1}^{0.5} N_5(\zeta)\,d\zeta = -\frac{81}{2048}; \quad \int_{-1}^{0.5} N_8(\zeta)\,d\zeta = \frac{1377}{35\,840};$$

$$\int_{-1}^{0.5} N_3(\zeta)\,d\zeta = \frac{12\,717}{1\,146\,880}; \quad \int_{-1}^{0.5} N_6(\zeta)\,d\zeta = -\frac{1377}{1\,146\,880}; \quad \int_{-1}^{0.5} N_9(\zeta)\,d\zeta = -\frac{81}{1\,146\,880};$$

$$\int_{-0.5}^{0.5} N_1(\zeta)\,d\zeta = \frac{1571}{53\,760}; \qquad \int_{-0.5}^{0.5} N_4(\zeta)\,d\zeta = \frac{1051}{143\,360}; \qquad \int_{-0.5}^{0.5} N_7(\zeta)\,d\zeta = \frac{683}{1\,290\,240};$$

$$\int_{-0.5}^{0.5} N_2(\zeta)\,d\zeta = \frac{25\,309}{26\,880}; \qquad \int_{-0.5}^{0.5} N_5(\zeta)\,d\zeta = 0; \qquad \int_{-0.5}^{0.5} N_8(\zeta)\,d\zeta = \frac{4201}{161\,280};$$

$$\int_{-0.5}^{0.5} N_3(\zeta)\,d\zeta = \frac{1571}{53\,760}; \qquad \int_{-0.5}^{0.5} N_6(\zeta)\,d\zeta = -\frac{1051}{143\,360}; \qquad \int_{-0.5}^{0.5} N_9(\zeta)\,d\zeta = \frac{683}{1\,290\,240};$$

$$\int_{-0.5}^{1} N_1(\zeta)\,d\zeta = \frac{12\,717}{1\,146\,880}; \qquad \int_{-0.5}^{1} N_4(\zeta)\,d\zeta = \frac{1377}{1\,146\,880}; \qquad \int_{-0.5}^{1} N_7(\zeta)\,d\zeta = -\frac{81}{1\,146\,880};$$

$$\int_{-0.5}^{1} N_2(\zeta)\,d\zeta = \frac{19\,359}{17\,920}; \qquad \int_{-0.5}^{1} N_5(\zeta)\,d\zeta = \frac{81}{2048}; \qquad \int_{-0.5}^{1} N_8(\zeta)\,d\zeta = \frac{1377}{35\,840};$$

$$\int_{-0.5}^{1} N_3(\zeta)\,d\zeta = \frac{468\,627}{1\,146\,880}; \qquad \int_{-0.5}^{1} N_6(\zeta)\,d\zeta = -\frac{72\,567}{1\,146\,880}; \qquad \int_{-0.5}^{1} N_9(\zeta)\,d\zeta = \frac{4329}{1\,146\,880}. \tag{10}$$

Collection of terms at f_i, f_i' and f_i'' in Eq. (6) results into a direct stiffness procedure [2] executed for all finite elements E_k, $k = 1, \ldots, 6$. Thus,

$$\begin{aligned}\int_{t_0}^{t_0+7h} f(t)\,dt = {} & h(a_1 f_1 + a_2 f_2 + a_3 f_3 + a_0 f_4 + a_0 f_5 + a_3 f_6 + a_2 f_7 + a_1 f_8) \\ & + h^2(b_1 f_1' + b_2 f_2' + b_3 f_3' + b_0 f_4' + b_0 f_5' - b_3 f_6' - b_2 f_7' - b_1 f_8') \\ & + h^3(c_1 f_1'' + c_2 f_2'' + c_3 f_3'' + c_0 f_4'' + c_0 f_5'' + c_3 f_6'' + c_2 f_7'' + c_1 f_8''),\end{aligned} \tag{11}$$

where the coefficients a_i, b_i and c_i, $i = 0, \ldots, 3$, are presented in Table 1.

It can be noted that the produced coefficients a_i, b_i and c_i, $i = 0, \ldots, 3$, can be used in integration formulas with different numbers of time steps. For example, if the number of time steps is 8:

$$\begin{aligned}\int_{t_0}^{t_0+8h} f(t)\,dt = {} & h(a_1 f_1 + a_2 f_2 + a_3 f_3 + a_0 f_4 + a_0 f_5 + a_0 f_6 + a_3 f_7 + a_2 f_8 + a_1 f_9) \\ & + h^2(b_1 f_1' + b_2 f_2' + b_3 f_3' + b_0 f_4' + b_0 f_5' + b_0 f_6' - b_3 f_7' - b_2 f_8' - b_1 f_9') \\ & + h^3(c_1 f_1'' + c_2 f_2'' + c_3 f_3'' + c_0 f_4'' + c_0 f_5'' + c_0 f_6'' + c_3 f_7'' + c_2 f_8'' + c_1 f_9'').\end{aligned} \tag{12}$$

The minimum number of nodes required to integrate exactly polynomials of degree at most 8 (using the derived coefficients a_i, b_i and c_i, $i = 0, \ldots, 3$) is 6. Later we will prove that the derived integration rule will also integrate exactly polynomials of degree at most 9.

3. General case

Let us assume that the discrete values of a function f and its first two derivatives are given at time moments $t_0 + (i-1)\cdot h$, $i = 1, \ldots, n$:

$$f_i = f\big(t_0 + (i-1)\cdot h\big), \qquad f_i' = f'\big(t_0 + (i-1)\cdot h\big), \qquad f_i'' = f''\big(t_0 + (i-1)\cdot h\big). \tag{13}$$

Let us interpolate the nodal values of the function f in domain of integration using one-dimensional finite elements with m nodes. Let us locate the first finite element E_1 in such a way that the co-ordinate of its first node is t_0, co-ordinate of the second node is $t_0 + h$, co-ordinate of the mth node is $t_0 + (m-1)h$. The second finite element E_2

Table 1

i	1	2	3	4	5	6	7	8
Coefficients at f_i								
E_1	$\frac{468\,627}{1\,146\,880}$	$\frac{19\,359}{17\,920}$	$\frac{12\,717}{1\,146\,880}$					
E_2		$\frac{1571}{53\,760}$	$\frac{25\,309}{26\,880}$	$\frac{1571}{53\,760}$				
E_3			$\frac{1571}{53\,760}$	$\frac{25\,309}{26\,880}$	$\frac{1571}{53\,760}$			
E_4				$\frac{1571}{53\,760}$	$\frac{25\,309}{26\,880}$	$\frac{1571}{53\,760}$		
E_5					$\frac{1571}{53\,760}$	$\frac{25\,309}{26\,880}$	$\frac{1571}{53\,760}$	
E_6						$\frac{12\,717}{1\,146\,880}$	$\frac{19\,359}{17\,920}$	$\frac{468\,627}{1\,146\,880}$
$\sum$	$\frac{468\,627}{1\,146\,880}$	$\frac{233}{210}$	$\frac{3\,378\,247}{3\,440\,640}$	1	1	$\frac{3\,378\,247}{3\,440\,640}$	$\frac{233}{210}$	$\frac{468\,627}{1\,146\,880}$
	a_1	a_2	a_3	a_0	a_0	a_3	a_2	a_1
Coefficients at f'_i								
E_1	$\frac{72\,567}{1\,146\,880}$	$-\frac{81}{2048}$	$-\frac{1377}{1\,146\,880}$					
E_2		$\frac{1051}{143\,360}$	0	$-\frac{1051}{143\,360}$				
E_3			$\frac{1051}{143\,360}$	0	$-\frac{1051}{143\,360}$			
E_4				$\frac{1051}{143\,360}$	0	$-\frac{1051}{143\,360}$		
E_5					$\frac{1051}{143\,360}$	0	$-\frac{1051}{143\,360}$	
E_6						$\frac{1377}{1\,146\,880}$	$\frac{81}{2048}$	$-\frac{72\,567}{1\,146\,880}$
$\sum$	$\frac{72\,567}{1\,146\,880}$	$-\frac{4619}{143\,360}$	$\frac{7031}{1\,146\,880}$	0	0	$-\frac{7031}{1\,146\,880}$	$\frac{4619}{143\,360}$	$-\frac{72\,567}{1\,146\,880}$
	b_1	b_2	b_3	b_0	b_0	$-b_3$	$-b_2$	$-b_1$
Coefficients at f''_i								
E_1	$\frac{4329}{1\,146\,880}$	$\frac{1377}{35\,840}$	$-\frac{81}{1\,146\,880}$					
E_2		$\frac{683}{1\,290\,240}$	$\frac{4201}{161\,280}$	$\frac{683}{1\,290\,240}$				
E_3			$\frac{683}{1\,290\,240}$	$\frac{4201}{161\,280}$	$\frac{683}{1\,290\,240}$			
E_4				$\frac{683}{1\,290\,240}$	$\frac{4201}{161\,280}$	$\frac{683}{1\,290\,240}$		
E_5					$\frac{683}{1\,290\,240}$	$\frac{4201}{161\,280}$	$\frac{683}{1\,290\,240}$	
E_6						$-\frac{81}{1\,146\,880}$	$\frac{1377}{35\,840}$	$\frac{4329}{1\,146\,880}$
$\sum$	$\frac{4329}{1\,146\,880}$	$\frac{10\,051}{258\,048}$	$\frac{273\,599}{10\,321\,920}$	$\frac{1943}{71\,680}$	$\frac{1943}{71\,680}$	$\frac{273\,599}{10\,321\,920}$	$\frac{10\,051}{258\,048}$	$\frac{4329}{1\,146\,880}$
	c_1	c_2	c_3	c_0	c_0	c_3	c_2	c_1

is shifted with respect to E_1 by h—its first node is located at $t_0 + h$; second node at $t_0 + 2h$; last node at $t_0 + mh$. The nodal co-ordinates of the kth finite element E_k are as follows: $t_0 + (k-1)h$ (first node); $t_0 + kh$ (second node); $t_0 + (k+m-2)h$ (last node). The process is continued until the last finite element E_{n-m+1} is placed—its first node at $t_0 + (n-m)h$; second node at $t_0 + (n-m+1)h$ and finally the last node at $t_0 + (n-1)h$.

Let us assign local co-ordinate ζ for every finite element in such a way that the co-ordinate of the ith node ζ_i is:

$$\zeta_i = -1 + 2 \cdot \frac{i-1}{m-1}, \quad i = 1, \ldots, m. \tag{14}$$

We will construct the shape functions of a finite element as polynomials of $(3m-1)$st degree.

$$N_i(\zeta) = d_{i0} + d_{i1}\zeta + \cdots + d_{i(3m-1)}\zeta^{3m-1}; \quad i = 1, \ldots, 3m. \tag{15}$$

The first set of the shape functions satisfy the following nodal conditions:

$$N_i(\zeta_i) = 1; \quad N_i(\zeta_j) = 0; \quad N'_i(\zeta_i) = 0; \quad N''_i(\zeta_i) = 0; \quad i = 1, \ldots, m; \ j = 1, \ldots, m; \ j \neq i. \tag{16}$$

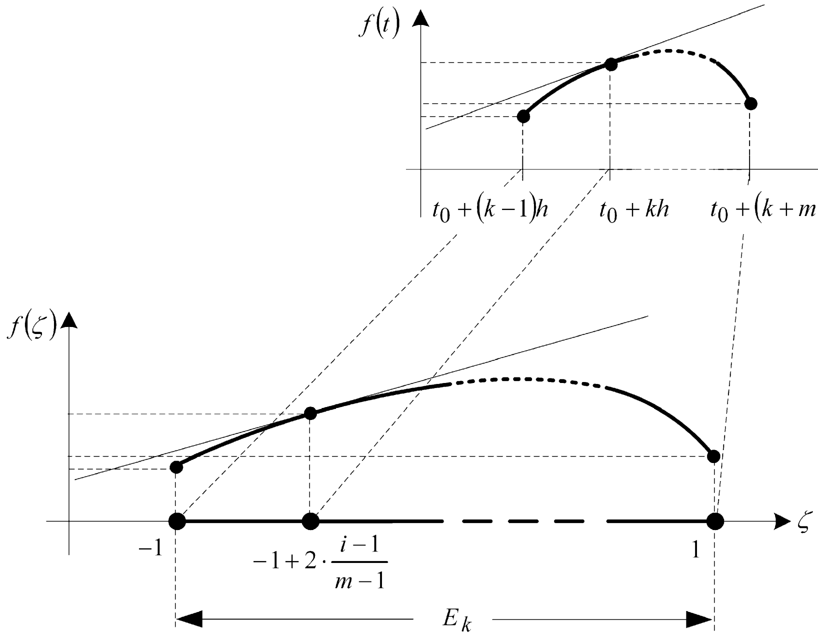

Fig. 2. Transformation of a time interval into the local domain of the kth finite element.

The second and the third sets of the shape functions satisfy such nodal conditions:

$$N_{m+i}(\zeta_i)=0;\quad N'_{m+i}(\zeta_i)=1;\quad N'_{m+i}(\zeta_j)=0;\quad N''_{m+i}(\zeta_i)=0;\quad i=1,\ldots,m;\ j=1,\ldots,m;\ j\neq i, \tag{17}$$

$$N_{2m+i}(\zeta_i)=0;\quad N'_{2m+i}(\zeta_i)=0;\quad N''_{2m+i}(\zeta_i)=1;\quad N''_{2m+i}(\zeta_j)=0;\quad i=1,\ldots,m;\ j=1,\ldots,m;\ j\neq i. \tag{18}$$

One and only one set of coefficients $d_{i0}, d_{i1}, \ldots, d_{i(3m-1)}$, $i = 1, \ldots, 3m$, in Eq. (15) can be determined from conditions in Eqs. (16)–(18).

It is clear that the relationship between the global co-ordinate t and the local co-ordinate $(-1 \leqslant \zeta \leqslant 1)$ of the kth finite element is:

$$t = t_0 + h\cdot\frac{m-1}{2}\cdot(\zeta+1) + h(k-1), \quad k = 1, \ldots, (n-m+1). \tag{19}$$

Then, the relationship between the nodal values of the function derivatives in the local and the global domain is (Fig. 2):

$$f(\zeta_i) = f_{i+k-1};\qquad \frac{df(\zeta_i)}{d\zeta} = \frac{h(m-1)}{2}\cdot f'_{i+k-1};\qquad \frac{d^2 f(\zeta_i)}{d\zeta^2} = \frac{h^2(m-1)^2}{4}\cdot f''_{i+k-1};\quad i = 1, \ldots, m. \tag{20}$$

A continuous function $\hat{f}(\zeta)$ can be interpolated in the domain of the kth finite element:

$$\hat{f}(\zeta) = \sum_{i=1}^{m}\left(f_{i+k-1}N_i(\zeta) + \frac{h(m-1)}{2}f'_{i+k-1}N_{m+i}(\zeta) + \frac{h^2(m-1)^2}{4}\cdot f''_{i+k-1}N_{2m+i}(\zeta)\right), \quad -1 \leqslant \zeta \leqslant 1. \tag{21}$$

It is clear that the function $\hat{f}(\zeta)$ is a polynomial of $(3m-1)$st degree satisfying the nodal conditions:

$$\hat{f}(\zeta_i) = f_{i+k-1};\qquad \frac{d\hat{f}(\zeta_i)}{d\zeta} = f'_{i+k-1};\qquad \frac{d^2\hat{f}(\zeta_i)}{d\zeta^2} = f''_{i+k-1}. \tag{22}$$

Let us split the definite integral of $f(t)$ into $n-m+1$ integrals:

$$\int_{t_0}^{t_0+(n-1)h} f(t)\,dt = \int_{t_0}^{t_0+\frac{hm}{2}} f(t)\,dt + \sum_{s=2}^{n-m}\left(\int_{t_0+\frac{hm}{2}+(s-2)h}^{t_0+\frac{hm}{2}+(s-1)h} f(t)\,dt\right) + \int_{t_0+\frac{hm}{2}+(n-m-1)h}^{t_0+(n-1)h} f(t)\,dt. \tag{23}$$

Introduction of relationship $t = t_0 + 0.5h(m-1)(\zeta+1)$ and assumption that the function $f(t)$ is a polynomial of the $(3m-1)$st degree leads to the following expression of the first integral on the right side of Eq. (23):

$$\int_{t_0}^{t_0+\frac{hm}{2}} f(t)\,dt = \frac{h(m-1)}{2}\int_{-1}^{\frac{1}{m-1}} f(\zeta)\,d\zeta = \frac{h(m-1)}{2}\sum_{i=1}^{m} f_i \int_{-1}^{\frac{1}{m-1}} N_i(\zeta)\,d\zeta$$
$$+ \frac{h^2(m-1)^2}{4}\sum_{i=1}^{m} f_i' \int_{-1}^{\frac{1}{m-1}} N_{m+i}(\zeta)\,d\zeta + \frac{h^3(m-1)^3}{8}\sum_{i=1}^{m} f_i'' \int_{-1}^{\frac{1}{m-1}} N_{2m+i}(\zeta)\,d\zeta. \tag{24}$$

Analogously, relationship $t = t_0 + h(s-1) + 0.5h(m-1)(\zeta+1)$ $(s = 2,\ldots,(n-m))$, yields:

$$\int_{t_0+\frac{hm}{2}+(s-2)h}^{t_0+\frac{hm}{2}+(s-1)h} f(t)\,dt = \frac{h(m-1)}{2}\int_{-\frac{1}{m-1}}^{\frac{1}{m-1}} f(\zeta)\,d\zeta = \frac{h(m-1)}{2}\sum_{i=1}^{m} f_{i+s-1}\int_{-\frac{1}{m-1}}^{\frac{1}{m-1}} N_i(\zeta)\,d\zeta$$
$$+ \frac{h^2(m-1)^2}{4}\sum_{i=1}^{m} f_{i+s-1}'\int_{-\frac{1}{m-1}}^{\frac{1}{m-1}} N_{m+i}(\zeta)\,d\zeta$$
$$+ \frac{h^3(m-1)^3}{8}\sum_{i=1}^{m} f_{i+s-1}''\int_{-\frac{1}{m-1}}^{\frac{1}{m-1}} N_{2m+i}(\zeta)\,d\zeta. \tag{25}$$

Finally, at $t = t_0 + h(n-m) + 0.5h(m-1)(\zeta+1)$:

$$\int_{t_0+\frac{hm}{2}+(n-m-1)h}^{t_0+(n-1)h} f(t)\,dt = \frac{h(m-1)}{2}\int_{-\frac{1}{m-1}}^{1} f(\zeta)\,d\zeta = \frac{h(m-1)}{2}\sum_{i=1}^{m} f_{i+n-m}\int_{-\frac{1}{m-1}}^{1} N_i(\zeta)\,d\zeta$$
$$+ \frac{h^2(m-1)^2}{4}\sum_{i=1}^{m} f_{i+n-m}'\int_{-\frac{1}{m-1}}^{1} N_{m+i}(\zeta)\,d\zeta$$
$$+ \frac{h^3(m-1)^3}{8}\sum_{i=1}^{m} f_{i+n-m}''\int_{-\frac{1}{m-1}}^{1} N_{2m+i}(\zeta)\,d\zeta. \tag{26}$$

Let us denote the integrals of the shape functions as:

$$n_i^L = \int_{-1}^{\frac{1}{m-1}} N_i(\zeta)\,d\zeta; \qquad n_i^C = \int_{-\frac{1}{m-1}}^{\frac{1}{m-1}} N_i(\zeta)\,d\zeta; \qquad n_i^R = \int_{-\frac{1}{m-1}}^{1} N_i(\zeta)\,d\zeta; \quad i = 1,\ldots,3m. \tag{27}$$

Eqs. (24)–(27) together with Eq. (23) yield:

$$\int_{t_0}^{t_0+(n-1)h} f(t)\,dt = \frac{h(m-1)}{2}\Bigg[f_1 n_1^L + f_2\big(n_2^L + n_1^C\big) + \cdots + f_m\Bigg(n_m^L + \sum_{i=1}^{m-1} n_{m-i}^C\Bigg) + \sum_{j=1}^{n-2m} f_{m+j}\sum_{i=1}^{m} n_i^C$$
$$+ f_{n-m+1}\Bigg(n_1^R + \sum_{i=1}^{m-1} n_{m-i+1}^C\Bigg) + \cdots + f_{n-1}\big(n_{m-1}^R + n_m^C\big) + f_n n_m^R\Bigg]$$
$$+ \frac{h^2(m-1)^2}{4}\Bigg[f_1' n_{m+1}^L + f_2'\big(n_{m+2}^L + n_{m+1}^C\big) + \cdots + f_m'\Bigg(n_{2m}^L + \sum_{i=1}^{m-1} n_{2m-i}^C\Bigg)$$
$$+ \sum_{j=1}^{n-2m} f_{m+j}' \sum_{i=1}^{m} n_{m+i}^C$$
$$+ f_{n-m+1}'\Bigg(n_{m+1}^R + \sum_{i=1}^{m-1} n_{2m-i+1}^C\Bigg) + \cdots + f_{n-1}'\big(n_{2m-1}^R + n_{2m}^C\big) + f_n' n_{2m}^R\Bigg]$$
$$+ \frac{h^3(m-1)^3}{8}\Bigg[f_1'' n_{2m+1}^L + f_2''\big(n_{2m+2}^L + n_{2m+1}^C\big) + \cdots + f_m''\Bigg(n_{3m}^L + \sum_{i=1}^{m-1} n_{3m-i}^C\Bigg)$$
$$+ \sum_{j=1}^{n-2m} f_{m+j}'' \sum_{i=1}^{m} n_{2m+i}^C$$
$$+ f_{n-m+1}''\Bigg(n_{2m+1}^R + \sum_{i=1}^{m-1} n_{3m-i+1}^C\Bigg) + \cdots + f_{n-1}''\big(n_{3m-1}^R + n_{3m}^C\big) + f_n'' n_{3m}^R\Bigg]. \tag{28}$$

It can be seen from the properties of the shape functions [2] that:

$$\begin{aligned}
&n_i^L = n_{m-i+1}^R, \qquad n_i^C = n_{m-i+1}^C;\\
&n_{m+i}^L = -n_{2m-i+1}^R, \qquad n_{m+i}^C = -n_{2m-i+1}^C;\\
&n_{2m+i}^L = n_{3m-i+1}^R, \qquad n_{2m+i}^C = n_{3m-i+1}^C;
\end{aligned} \tag{29}$$

$i = 1, \ldots, m$.

Therefore the coefficients at f_i and f_{m-i+1} are equal. The coefficients at f_i'' and f_{m-i+1}'' are equal too. The absolute values of coefficients at f_i' and f_{m-i+1}' are equal, but their signs are different.

Let us denote:

$$\begin{aligned}
&a_1 = \frac{m-1}{2} n_1^L; \qquad b_1 = \frac{(m-1)^2}{4} n_{m+1}^L; \qquad c_1 = \frac{(m-1)^3}{8} n_{2m+1}^L;\\
&a_2 = \frac{m-1}{2}\big(n_2^L + n_1^C\big); \qquad b_2 = \frac{(m-1)^2}{4}\big(n_{m+2}^L + n_{m+1}^C\big); \qquad c_2 = \frac{(m-1)^3}{8}\big(n_{2m+2}^L + n_{2m+1}^C\big);\\
&\quad\vdots\\
&a_m = \frac{m-1}{2}\Bigg(n_m^L + \sum_{i=1}^{m-1} n_{m-i}^C\Bigg); \qquad b_m = \frac{(m-1)^2}{4}\Bigg(n_{2m}^L + \sum_{i=1}^{m-1} n_{2m-i}^C\Bigg);\\
&c_m = \frac{(m-1)^3}{8}\Bigg(n_{3m}^L + \sum_{i=1}^{m-1} n_{3m-i}^C\Bigg);\\
&a_0 = \frac{m-1}{2}\sum_{i=1}^{m} n_i^C; \qquad b_0 = \frac{(m-1)^2}{4}\sum_{i=1}^{m} n_{m+i}^C; \qquad c_0 = \frac{(m-1)^3}{8}\sum_{i=1}^{m} n_{2m+i}^C.
\end{aligned} \tag{30}$$

Then,

$$\int_{t_0}^{t_0+(n-1)h} f(t)\,dt = \left(a_1 f_1 + a_2 f_2 + \cdots + a_m f_m + \sum_{i=1}^{n-2m} a_0 f_{m+i} + a_m f_{n-m+1} + \cdots + a_2 f_{n-1} + a_1 f_n\right)\cdot h$$
$$+ \left(b_1 f_1' + b_2 f_2' + \cdots + b_m f_m' + \sum_{i=1}^{n-2m} b_0 f_{m+i}' - b_m f_{n-m+1}' - \cdots - b_2 f_{n-1}' - b_1 f_n'\right)\cdot h^2$$
$$+ \left(c_1 f_1'' + c_2 f_2'' + \cdots + c_m f_m'' + \sum_{i=1}^{n-2m} c_0 f_{m+i}'' + c_m f_{n-m+1}'' + \cdots + c_2 f_{n-1}'' + c_1 f_n''\right)\cdot h^3. \tag{31}$$

4. Properties of the weight coefficients

Let us assume that $f(t) = C = \text{const}$ and $n > 2m$. Thus, $f_i = C$, $f_i' = 0$, $f_i'' = 0$, $i = 1, \ldots, n$. It is clear that,

$$C\cdot\int_{t_0}^{t_0+nh} dt - C\cdot\int_{t_0}^{t_0+(n-1)h} dt = hC. \tag{32}$$

From Eq. (31) it follows that:

$$C\cdot\int_{t_0}^{t_0+nh} dt - C\cdot\int_{t_0}^{t_0+(n-1)h} dt = hC\left(2\sum_{i=1}^{m} a_i + a_0(n-2m+1)\right) - hC\left(2\sum_{i=1}^{m} a_i + a_0(n-2m)\right) = a_0 hC. \tag{33}$$

Comparison of the right sides of Eqs. (32) and (33) produces equality $a_0 = 1$. Then, from Eq. (31) it follows that

$$C\cdot\int_{t_0}^{t_0+(n-1)h} dt = hC\left(2\sum_{i=1}^{m} a_i + n - 2m\right) = hC(n-1), \tag{34}$$

what leads to the equality:

$$\sum_{i=1}^{m} a_i = \frac{2m-1}{2}. \tag{35}$$

Let us assume that $f(t) = Ct$, $C = \text{const}$ and $n > 2m$. Thus, $f_i = C(t_0 + (i-1)h)$, $f_i' = C$, $f_i'' = 0$, $i = 1, \ldots, n$.
Then, analogously,

$$C\cdot\int_{t_0}^{t_0+nh} t\,dt - C\cdot\int_{t_0}^{t_0+(n-1)h} t\,dt = \frac{C}{2}\left((t_0+nh)^2 - \left(t_0+(n-1)h\right)^2\right) = Ch\left(t_0 + \left(n - \frac{1}{2}\right)h\right); \tag{36}$$

$$C\cdot\int_{t_0}^{t_0+nh} t\,dt - C\cdot\int_{t_0}^{t_0+(n-1)h} t\,dt = h\left(\sum_{i=1}^{n-2m+1} f_{m+i} - \sum_{i=1}^{n-2m} f_{m+i}\right)$$
$$+ h\left(\sum_{i=1}^{m} a_{m-i+1}\cdot f_{n-m+i+1} - \sum_{i=1}^{m} a_{m-i+1}\cdot f_{n-m+i}\right)$$
$$+ b_0 h^2\left(\sum_{i=1}^{n-2m+1} f_{m+i}' - \sum_{i=1}^{n-2m} f_{m+i}'\right)$$
$$= Ch\left(t_0 + (n-m)h + h\sum_{i=1}^{m} a_{m-i+1} + b_0 h\right). \tag{37}$$

Comparison of the right sides of equalities in Eqs. (36) and (37) produces:

$$\sum_{i=1}^{m} a_{m-i+1} + b_0 = \frac{2m-1}{2}, \tag{38}$$

what results into the following condition:

$$b_0 = 0. \tag{39}$$

Now, let us assume that $f(t) = \frac{C}{2} \cdot t^2$. Then, $f_i = \frac{C}{2}(t_0 + (i-1)h)^2$, $f_i' = C(t_0 + (i-1)h)$, $f_i'' = C$, $i = 1, \dots, n$. Analogously,

$$\frac{C}{2} \int_{t_0}^{t_0+nh} t^2\,dt - \frac{C}{2} \int_{t_0}^{t_0+(n-1)h} t^2\,dt = \frac{C}{6}\big((t_0+nh)^3 - (t_0+(n-1)h)^3\big); \tag{40}$$

$$\frac{C}{2} \int_{t_0}^{t_0+nh} t^2\,dt - \frac{C}{2} \int_{t_0}^{t_0+(n-1)h} t^2\,dt = h \cdot f_{n-m+1} + h \cdot \sum_{i=1}^{m} a_{m-i+1} \cdot (f_{n-m+i+1} - f_{n-m+i})$$
$$- h^2 \cdot \sum_{i=1}^{m} b_{m-i+1} \cdot (f'_{n-m+i} - f'_{n-m+i+1}) + h^3 \cdot c_0 \cdot f''_{n-m+1}. \tag{41}$$

Thus,

$$c_0 = \sum_{i=1}^{m} (b_i - i \cdot a_{m-i+1}) + \frac{6m^2-1}{12}. \tag{42}$$

Finally, the integration rule in Eq. (31) can be simplified to the following form:

$$\int_{t_0}^{t_0+(n-1)h} f(t)\,dt = \left(\sum_{i=1}^{m} a_i f_i + \sum_{i=1}^{n-2m} f_{m+i} + \sum_{i=1}^{m} a_{m-i+1} f_{n-m+i}\right)h + \left(\sum_{i=1}^{m} b_i f_i' - \sum_{i=1}^{m} b_{m-i+1} f'_{n-m+i}\right)h^2$$
$$+ \left(\sum_{i=1}^{m} c_i f_i'' + \sum_{i=1}^{n-2m} c_0 f''_{m+i} + \sum_{i=1}^{m} c_{m-i+1} f''_{n-m+i}\right)h^3. \tag{43}$$

It is clear that the derived integration rule is exact when the integrated function is a polynomial of degree at most $(3m-1)$. We will prove that this integration rule is exact when the integrated function is a polynomial of degree at most $3m$, if only m is odd.

The numerical values of the coefficients in the integration rule (43) are presented at different values of m in Table 4. The parameter p in these tables denotes the maximum degree of exactly integrated polynomials.

Theorem. *With m even the integration rule given in* (43) *is exact when $f(t)$ is a polynomial of degree at most* $(3m-1)$. *With m odd the integration rule given in* (43) *is exact when $f(t)$ is a polynomial of degree at most* $3m$.

Proof. One and only one polynomial of the $(3m-1)$st degree can be interpolated in the domain of the ith finite element through the discrete values of the function f and its derivatives (Fig. 3):

$$\hat{f}_i(t) = r_{i,0} + r_{i,1}t + \cdots + r_{i,(3m-1)}t^{(3m-1)}. \tag{44}$$

Many different polynomials of the $3m$th degree can be interpolated through the same points:

$$P_i(t) = s_{i,0} + s_{i,1}t + \cdots + s_{i,3m}t^{3m}. \tag{45}$$

Coefficients r and s in Eqs. (44) and (45) are real numbers. Let us calculate the difference between the polynomials in Eqs. (44) and (45):

$$\Delta_i(t) = P_i(t) - \hat{f}_i(t). \tag{46}$$

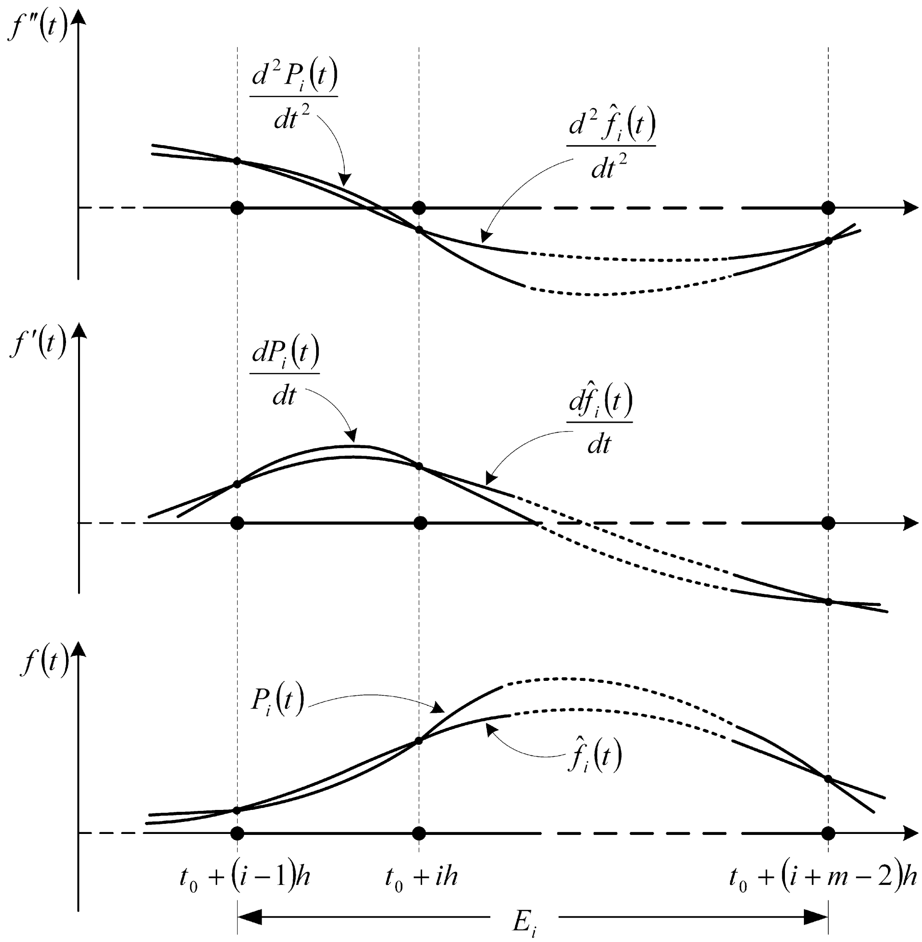

Fig. 3. Interpolation in the domain of finite element E_i.

Then, the derivatives $\Delta_i'(t)$ and $\Delta_i''(t)$ will be:

$$\Delta_i'(t) = P_i'(t) - \hat{f}_i'(t); \qquad \Delta_i''(t) = P_i''(t) - \hat{f}_i''(t). \tag{47}$$

Both polynomials $\hat{f}_i(t)$ and $P_i(t)$ do not only interpolate the function values, but also satisfy the nodal conditions for derivatives. Thus the abscises on the nodal points will be the roots of polynomials $\Delta_i(t)$, $\Delta_i'(t)$ and $\Delta_i''(t)$:

$$\begin{aligned} &\Delta_i(t_0 + jh) = 0, \quad j = (i-1), i, \ldots, (i+m-2); \\ &\Delta_i'(t_0 + jh) = 0, \quad j = (i-1), i, \ldots, (i+m-2); \\ &\Delta_i''(t_0 + jh) = 0, \quad j = (i-1), i, \ldots, (i+m-2). \end{aligned} \tag{48}$$

Keeping in mind the result in Eq. (48) and in accordance to the Viete theorem [9], polynomial $\Delta_i(t)$ can be expressed like:

$$\Delta_i(t) = s_{i,m}\big[t - t_0 - (i-1)h\big]^3 [t - t_0 - ih]^3 \cdots \big[t - t_0 - (i+m-2)h\big]^3. \tag{49}$$

Introduction of a new variable $\zeta = \frac{2}{h(m-1)}(t - t_0 - (i + \frac{m-3}{2})h)$ helps to simplify the expression of $\Delta_i(t)$:

$$\Delta_i(\zeta) = s_{i,3m}(\zeta+1)^3\left(\zeta + \frac{m-3}{m-1}\right)^3 \cdots \left(\zeta - \frac{m-3}{m-1}\right)^3 (\zeta-1)^3. \tag{50}$$

Then the integral of $\Delta_i(t)$ in the domain of the ith finite element is:

$$\int_{t_0+(i-1)h}^{t_0+(i+m-2)h} \Delta_i(t)\,dt = \frac{h(m-1)}{2}\int_{-1}^{1} \Delta_i(\zeta)\,d\zeta. \tag{51}$$

Let us assume that m is odd. Then the integral in Eq. (51) is equal to zero because the integrand is an odd function and the limits of the integral are symmetric around the origin. The limits of the integrals in the domains of the internal finite elements E_i, $i = 2, 3, \ldots, (n-m)$, are $-\frac{1}{m-1} \leqslant \zeta \leqslant \frac{1}{m-1}$ (Eq. (25)). Therefore these integrals are also equal to zero due to the symmetry of limits.

The limits of integration for the first finite element are $-1 \leqslant \zeta \leqslant \frac{1}{m-1}$. The limits for the last finite element are $-\frac{1}{m-1} \leqslant \zeta \leqslant 1$. But the domains of integration for the first and the last finite elements can be simplified:

$$\int_{-1}^{\frac{1}{m-1}} \Delta_1(\zeta)\,d\zeta + \int_{-\frac{1}{m-1}}^{1} \Delta_{n-m+1}(\zeta)\,d\zeta = \int_{-1}^{-\frac{1}{m-1}} \Delta_1(\zeta)\,d\zeta + \int_{\frac{1}{m-1}}^{1} \Delta_{n-m+1}(\zeta)\,d\zeta. \tag{52}$$

Let us assume that the integrated function $f(t)$ is a polynomial of degree at most $3m$. Then the coefficients s in Eq. (45) do not depend from i. Particularly,

$$s_{1,3m} = s_{(n-m+1),3m}. \tag{53}$$

Then the sum in Eq. (52) is equal to zero due to the symmetry of the limits. Finally,

$$\int_{-1}^{\frac{1}{m-1}} \Delta_1(\zeta)\,d\zeta + \sum_{i=2}^{n-m} \int_{-\frac{1}{m-1}}^{\frac{1}{m-1}} \Delta_i(\zeta)\,d\zeta + \int_{-\frac{1}{m-1}}^{1} \Delta_{n-m+1}(\zeta)\,d\zeta = 0. \tag{54}$$

Thus the derived integration rule is exact when the integrated function is a polynomial of degree at most $3m$ and m is odd.

If m is even, the function $\Delta_i(t)$ in Eq. (46) is even and its integral is not equal to zero though the integration limits are symmetric around the origin. In that case the integration rule is exact only when the integrated function is a polynomial of degree at most $(3m-1)$. □

Algorithm. The derived integration rule (Eq. (43)) can be presented in the form of the following algorithm.

0. Define the time step h. Select the parameter m (the degree of exactly integrated polynomial). Initialise $Sum_1^{(k)} = 0$, $Sum_2^{(k)} = 0$, $Sum_3^{(k)} = 0$ $(k = 2, \ldots, m)$ and $i = 0$. Allocate memory for $3(m-1)$ queues of k elements: $Q_1^{(k)}$, $Q_2^{(k)}$ and $Q_3^{(k)}$, $k = 2, \ldots, m$.
1. Get a new set of function values (f, f', f''). If a new set of function values is available—go to Step 2. Otherwise go to Step 5.
2. Increment counter $i = i + 1$; save current values f, f', f'' in the queues $Q_1^{(k)}$, $Q_2^{(k)}$, $Q_3^{(k)}$, $k = 2, \ldots, m$, accordingly. When saving an element in a queue, the queue's cells are shifted, the last element is deleted and the new value is stored in the first element of the queue.
3. Repeat for all k from 2 to m
   If $i > k$, then

$$Sum_1^{(k)} = Sum_1^{(k)} + f;$$

$$Sum_3^{(k)} = Sum_3^{(k)} + c^{(k)}(0) \cdot f'';$$

Otherwise

$$Sum_1^{(k)} = Sum_1^{(k)} + a^{(k)}(i) \cdot f;$$
$$Sum_2^{(k)} = Sum_2^{(k)} + b^{(k)}(i) \cdot f';$$
$$Sum_3^{(k)} = Sum_3^{(k)} + c^{(k)}(i) \cdot f''.$$

4. Go to Step 1.
5. Repeat for all k from 2 to m
   If $i \geqslant 2k$, then repeat for all j from 1 to k:
   $$Sum_1^{(k)} = Sum_1^{(k)} + \left(a^{(k)}(j) - 1\right) \cdot Q_1^{(k)}(j);$$
   $$Sum_2^{(k)} = Sum_2^{(k)} - b^{(k)}(j) \cdot Q_2^{(k)}(j);$$
   $$Sum_3^{(k)} = Sum_3^{(k)} + \left(c^{(k)}(j) - c^{(k)}(0)\right) \cdot Q_3^{(k)}(j).$$
6. Repeat for all k from 2 to m:
   If $i \geqslant 2k$, then calculate and display the integral:
   $$I^{(k)} = Sum_1^{(k)} \cdot h + Sum_2^{(k)} \cdot h^2 + Sum_3^{(k)} \cdot h^3.$$
   Otherwise display warning message “Data series is too short for degree k”.
   and STOP.

5. Error estimates of the integration rule

The sampled function f is interpolated in the domain of the kth finite element as a polynomial $\hat{f}$ of degree $(3m-1)$.

Let us assume that the integrand has at least $3m$ bounded derivatives in this domain. Then the error of interpolation in the domain of the kth finite element is calculated as:

$$\begin{aligned} e_k(t) &= f_k(t) - \hat{f}_k(t) \\ &= f_k\big[t, \underbrace{t_0+(k-1)h, t_0+(k-1)h, t_0+(k-1)h}_{3}, \ldots, \\ &\qquad \underbrace{t_0+(k+m-2)h, t_0+(k+m-2)h, t_0+(k+m-2)h}_{3}\big] \cdot \omega(t), \end{aligned} \tag{55}$$

where

$$f_k[t, \underbrace{t_0+(k-1)h, t_0+(k-1)h, t_0+(k-1)h}_{3}, \ldots, \underbrace{t_0+(k+m-2)h, t_0+(k+m-2)h, t_0+(k+m-2)h}_{3}]$$

is the $3m$th divided difference in the kth finite element; $\omega(t) = (t - t_0 - (k-1)h)^3 (t - t_0 - kh)^3 \cdots (t - t_0 - (k + m - 2)h)^3$. It must be noted that each node of the finite element is repeated thrice in the divided difference.

Compound error of the derived quadrature rule (Eq. (23)) takes the following form:

$$R_{3m} = \int_{t_0}^{t_0+\frac{hm}{2}} e_1(t)\,dt + \sum_{s=2}^{n-m} \left(\int_{t_0+\frac{hm}{2}+(s-2)h}^{t_0+\frac{hm}{2}+(s-1)h} e_s(t)\,dt \right) + \int_{t_0+\frac{hm}{2}+(n-m-1)h}^{t_0+(n-1)h} e_{n-m+1}(t)\,dt. \tag{56}$$

Introduction of a new variable $\zeta = \frac{2}{h(m-1)}(t - t_0 - (k + \frac{m-3}{2})h)$ helps to simplify the expression in Eq. (56):

$$R_{3m} = \frac{h(m-1)}{2} \cdot \left(\int_{-1}^{\frac{1}{m-1}} e_1(\zeta)\,d\zeta + \sum_{s=2}^{n-m} \left(\int_{-\frac{1}{m-1}}^{\frac{1}{m-1}} e_s(\zeta)\,d\zeta \right) + \int_{-\frac{1}{m-1}}^{1} e_{n-m+1}(\zeta)\,d\zeta \right), \tag{57}$$

where $e_k(\zeta) = f_k[\zeta, -1, -1, -1, \ldots, 1, 1, 1] \cdot \omega^*(\zeta)$; $\omega^*(\zeta) = (\zeta+1)^3(\zeta + \frac{m-3}{m-1})^3 \cdots (\zeta - \frac{m-3}{m-1})^3(\zeta-1)^3$.

The integrals on the right side of Eq. (57) are evaluated as follows:

1. If m is odd,

$$\int_{-\frac{1}{m-1}}^{\frac{1}{m-1}} e_k(\zeta)\,d\zeta = \frac{f^{(3m+1)}(c)}{(3m+1)!} \cdot \int_{-\frac{1}{m-1}}^{\frac{1}{m-1}} \zeta \cdot \omega^*(\zeta)\,d\zeta, \tag{58}$$

$$\int_{-1}^{\frac{1}{m-1}} e_1(\zeta)\,d\zeta = \int_{-1}^{-\frac{1}{m-1}} e_1(\zeta)\,d\zeta + \int_{-\frac{1}{m-1}}^{\frac{1}{m-1}} e_1(\zeta)\,d\zeta$$
$$= \frac{f^{(3m)}(c_1)}{(3m)!} \cdot \int_{-1}^{-\frac{1}{m-1}} \omega^*(\zeta)\,d\zeta + \frac{f^{(3m+1)}(c_2)}{(3m+1)!} \cdot \int_{-\frac{1}{m-1}}^{\frac{1}{m-1}} \zeta \cdot \omega^*(\zeta)\,d\zeta, \tag{59}$$

$$\int_{-\frac{1}{m-1}}^{1} e_{n-m+1}(\zeta)\,d\zeta = \int_{\frac{1}{m-1}}^{1} e_{n-m+1}(\zeta)\,d\zeta + \int_{-\frac{1}{m-1}}^{\frac{1}{m-1}} e_{n-m+1}(\zeta)\,d\zeta$$
$$= \frac{f^{(3m)}(c_1)}{(3m)!} \cdot \int_{\frac{1}{m-1}}^{1} \omega^*(\zeta)\,d\zeta + \frac{f^{(3m+1)}(c_2)}{(3m+1)!} \cdot \int_{-\frac{1}{m-1}}^{\frac{1}{m-1}} \zeta \cdot \omega^*(\zeta)\,d\zeta. \tag{60}$$

2. If m is even,

$$\int_{-\frac{1}{m-1}}^{\frac{1}{m-1}} e_k(\zeta)\,d\zeta = \frac{f^{(3m)}(c)}{(3m)!} \cdot \int_{-\frac{1}{m-1}}^{\frac{1}{m-1}} \omega^*(\zeta)\,d\zeta, \tag{61}$$

$$\int_{-1}^{\frac{1}{m-1}} e_1(\zeta)\,d\zeta = \int_{-1}^{-\frac{1}{m-1}} e_1(\zeta)\,d\zeta + \int_{-\frac{1}{m-1}}^{\frac{1}{m-1}} e_1(\zeta)\,d\zeta = \frac{f^{(3m)}(c)}{(3m)!} \cdot \left[\int_{-1}^{-\frac{1}{m-1}} \omega^*(\zeta)\,d\zeta + \int_{-\frac{1}{m-1}}^{\frac{1}{m-1}} \omega^*(\zeta)\,d\zeta \right], \tag{62}$$

$$\int_{-\frac{1}{m-1}}^{1} e_{n-m+1}(\zeta)\,d\zeta = \int_{\frac{1}{m-1}}^{1} e_{n-m+1}(\zeta)\,d\zeta + \int_{-\frac{1}{m-1}}^{\frac{1}{m-1}} e_{n-m+1}(\zeta)\,d\zeta$$
$$= \frac{f^{(3m)}(c)}{(3m)!} \cdot \left[\int_{\frac{1}{m-1}}^{1} \omega^*(\zeta)\,d\zeta + \int_{-\frac{1}{m-1}}^{\frac{1}{m-1}} \omega^*(\zeta)\,d\zeta \right], \tag{63}$$

where the $3m$th and $(3m+1)$th derivatives are evaluated at any point in the local domain of every finite element ($c, c_1, c_2 \in [-1; 1]$). These evaluations are produced applying analogous techniques used for the derivation of error terms in Newton–Cotes quadrature rules [6].

Finally, Eqs. (57)–(63) lead to the following estimations:

1. If m is odd, the estimation of the compound error term of the derived integration rule:
$$|R_{3m}| \leqslant \frac{1}{(3m+1)!}\left(\frac{h(m-1)}{2}\right)^{3m+2} \cdot \left(M_1 \frac{4(3m+1)}{h(m-1)} I_1 + M_2(n-m+1)I_2\right), \tag{64}$$
where $M_1 = \sup_x |f^{(3m)}(x)|$, $M_2 = \sup_x |f^{(3m+1)}(x)|$, $x \in [t_0; t_0 + (n-1)h]$.
2. If m is even, the estimation of the compound error term of the derived integration rule is:
$$|R_{3m}| \leqslant \frac{M}{(3m)!}\left(\frac{h(m-1)}{2}\right)^{3m+1} \cdot \big(2I_1 + (n-m+1)I_3\big), \tag{65}$$
where $M = \sup_x |f^{(3m)}(x)|$, $x \in [t_0; t_0 + (n-1)h]$.

Integrals I_1, I_2 and I_3 in Eqs. (64) and (65) can be calculated as follows:
$$I_1 = \left|\int_{-1}^{-\frac{1}{m-1}} \omega^*(\zeta)\, d\zeta\right| = \left|\int_{\frac{1}{m-1}}^{1} \omega^*(\zeta)\, d\zeta\right|, \qquad I_2 = \left|\int_{-\frac{1}{m-1}}^{\frac{1}{m-1}} \zeta \cdot \omega^*(\zeta)\, d\zeta\right|, \qquad I_3 = \left|\int_{-\frac{1}{m-1}}^{\frac{1}{m-1}} \omega^*(\zeta)\, d\zeta\right|. \tag{66}$$

It can be noted that the complementary factors $(h(m-1)/2)^{3m+1}$ and $(h(m-1)/2)^{3m}$ in the first and the second error estimations follow from the relationship between the nodal values of the function derivatives in the local and the global domain (Fig. 2).

It is clear that the number of nodes n is proportional to $1/h$. For this reason the error term of the derived integration rule is of order h^{3m+1} if m is odd and of order h^{3m} if m is even.

Compound error estimations for several values of m are presented below:

$$m=2: \quad |R_6| \leqslant M \cdot h^7 \frac{1}{100\,800}(n-1), \quad \text{where } M = \sup_x \big|f^{(6)}(x)\big|.$$
$$m=3: \quad |R_9| \leqslant h^{10}\left(M_1 \frac{1}{11\,468\,800} + M_2 h \frac{8299}{4\,291\,854\,336\,000}(n-2)\right),$$
$$\text{where } M_1 = \sup_x \big|f^{(9)}(x)\big|,\ M_2 = \sup_x \big|f^{(10)}(x)\big|.$$
$$m=4: \quad |R_{12}| \leqslant M \cdot h^{13}\left(\frac{3617}{2\,054\,916\,864\,000} + \frac{673}{4\,109\,833\,728\,000}(n-3)\right), \quad \text{where } M = \sup_x \big|f^{(12)}(x)\big|.$$
$$m=5: \quad |R_{15}| \leqslant h^{16}\left(M_1 \frac{1515}{52\,665\,962\,725\,376} + M_2 h \frac{6\,610\,230\,619}{350\,004\,624\,284\,807\,331\,840\,000}(n-4)\right),$$
$$\text{where } M_1 = \sup_x \big|f^{(15)}(x)\big|,\ M_2 = \sup_x \big|f^{(16)}(x)\big|.$$

6. Concluding remarks

The derived integration rule is a generalisation of the trapezoidal rule both in the sense of the degree of exactly integrated polynomial and in the sense of the number of discrete values of derivatives of the integrand at every node. Particularly, we demonstrated the derivation of the rule for three nodal values—f_i, f_i' and f_i''. It has been shown that the degree of precision of this rule is increased by one, only if the parameter m is odd. Natural is the interest how the integration rule would look like if the number of nodal values of the integrand is different. When this number is one (only f_i), the integration rule is presented in Table 2. It can be noted that the degree of precision of this rule is also increased by one if m is odd. What would happen when two nodal parameters (f_i and f_i') are given at every node?

In general, the derivation of such integration rule is analogous to the one presented in this paper. The weights of such integration rule are presented in Table 3. Nevertheless there is a substantial difference in regards to the degree

Table 2
Nodal weights of the integration rule with nodal values of f_i

m	2	3	4	5	6	7
a_0	1	1	1	1	1	1
a_1	$\frac{1}{2}$	$\frac{3}{8}$	$\frac{1}{3}$	$\frac{95}{288}$	$\frac{51}{160}$	$\frac{5257}{17280}$
a_2	1	$\frac{7}{6}$	$\frac{31}{24}$	$\frac{317}{240}$	$\frac{991}{720}$	$\frac{22081}{15120}$
a_3		$\frac{23}{24}$	$\frac{5}{6}$	$\frac{23}{30}$	$\frac{59}{90}$	$\frac{54851}{120960}$
a_4			$\frac{25}{24}$	$\frac{793}{720}$	$\frac{97}{80}$	$\frac{103}{70}$
a_5				$\frac{157}{160}$	$\frac{1333}{1440}$	$\frac{89437}{120960}$
a_6					$\frac{91}{90}$	$\frac{16367}{15120}$
a_7						$\frac{23917}{24192}$
$\sum_{i=1}^{m} a_i$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$	$\frac{9}{2}$	$\frac{11}{2}$	$\frac{13}{2}$
p	1	3	3	5	5	7
l	2	4	4	6	6	8

Table 3
Nodal weights of the integration rule with nodal values of f_i and f_i'

m	2	3	4	5	6	7
a_0	1	1	1	1	1	1
a_1	$\frac{1}{2}$	$\frac{1131}{2560}$	$\frac{223}{567}$	$\frac{161\,002\,985}{445\,906\,944}$	$\frac{477\,033}{1\,408\,000}$	$\frac{10\,686\,787\,637\,771}{33\,124\,515\,840\,000}$
a_2	1	$\frac{31}{30}$	$\frac{649}{672}$	$\frac{2\,075\,083}{2\,580\,480}$	$\frac{164\,837}{285\,120}$	$\frac{1\,560\,978\,733\,133}{5\,101\,175\,439\,360}$
a_3		$\frac{7871}{7680}$	$\frac{47}{42}$	$\frac{731}{630}$	$\frac{8527}{8910}$	$\frac{6\,096\,817\,066\,859}{20\,404\,701\,757\,440}$
a_4			$\frac{18\,541}{18\,144}$	$\frac{11\,536\,369}{9\,953\,280}$	$\frac{5051}{3520}$	$\frac{15\,549}{10\,010}$
a_5				$\frac{9\,316\,481}{9\,175\,040}$	$\frac{5\,383\,903}{4\,561\,920}$	$\frac{3\,369\,006\,926\,303}{1\,854\,972\,887\,040}$
a_6					$\frac{450\,367}{445\,500}$	$\frac{761\,963\,150\,017\,127}{637\,646\,929\,920\,000}$
a_7						$\frac{373\,987\,201\,123}{370\,994\,577\,408}$
$\sum_{i=1}^{m} a_i$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$	$\frac{9}{2}$	$\frac{11}{2}$	$\frac{13}{2}$
p	3	5	7	9	11	13
b_0	0	0	0	0	0	0
b_1	$\frac{1}{12}$	$\frac{153}{2560}$	$\frac{43}{945}$	$\frac{2\,783\,825}{74\,317\,824}$	$\frac{31\,989}{985\,600}$	$\frac{1\,060\,070\,310\,089}{36\,436\,967\,424\,000}$
b_2	0	$-\frac{101}{1920}$	$-\frac{43}{288}$	$-\frac{665\,683}{2\,580\,480}$	$-\frac{11\,683}{31\,680}$	$-\frac{37\,023\,276\,497}{77\,290\,536\,960}$
b_3		$-\frac{53}{7680}$	$-\frac{97}{1260}$	$-\frac{2\,879\,083}{10\,321\,920}$	$-\frac{886\,799}{1\,330\,560}$	$-\frac{4\,380\,729\,864\,067}{3\,400\,783\,626\,240}$
b_4			$-\frac{163}{30\,240}$	$-\frac{1\,796\,843}{23\,224\,320}$	$-\frac{30\,343}{73\,920}$	$-\frac{890\,704\,364\,161}{637\,646\,929\,920}$
b_5				$-\frac{15\,867}{4\,587\,520}$	$-\frac{390\,869}{5\,322\,240}$	$-\frac{146\,993\,461\,711}{261\,598\,740\,480}$
b_6					$-\frac{79}{34\,650}$	$-\frac{1\,485\,451\,216\,771}{21\,254\,897\,664\,000}$
b_7						$-\frac{3\,238\,339\,925}{2\,040\,470\,175\,744}$
$\sum_{i=1}^{m} b_i$	$\frac{1}{12}$	$\frac{1}{3\,840}$	$-\frac{563}{3\,024}$	$-\frac{3\,080\,299}{5\,308\,416}$	$-\frac{2\,830\,321}{1\,900\,800}$	$-\frac{1\,040\,196\,577\,921}{276\,037\,632\,000}$
p	3	5	7	9	11	13
l	4	6	8	10	12	14

of precision of the rule—it is not increased by one neither when m is odd nor even. That is because the difference between the polynomials $P_i(t)$ and $\hat{f}_i(t)$ (Eq. (46)) is an even function and therefore its integral is not equal to zero.

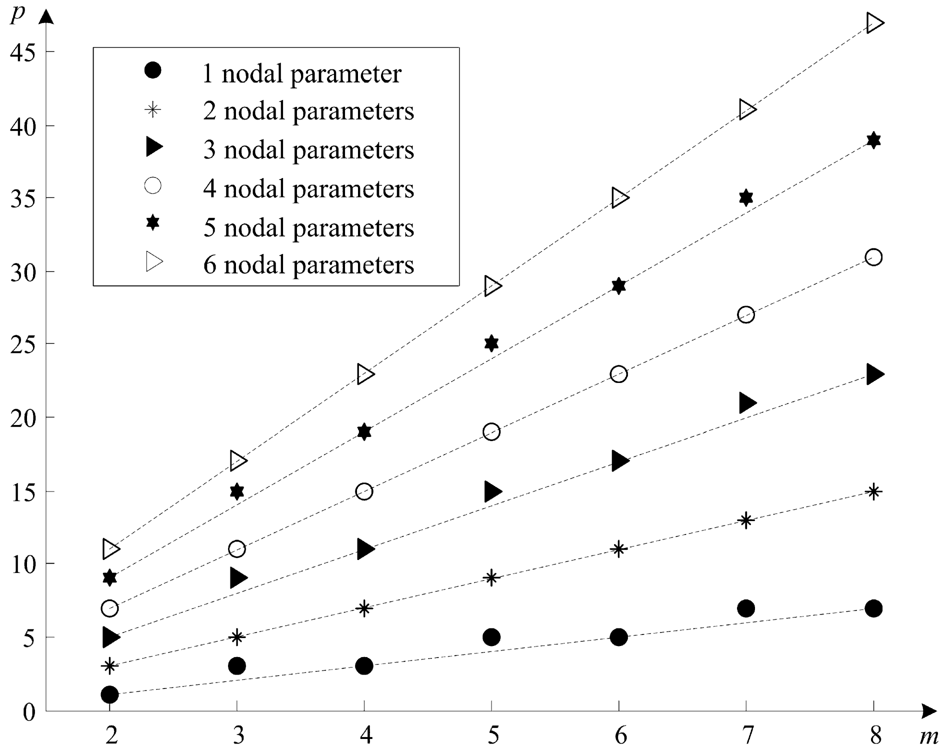

Fig. 4. Degrees of precision of the derived integration rule for different number of nodal derivatives.

Moreover, such analysis can be extended to larger numbers of nodal parameters (derivatives of the integrand), and the rule's degrees of precision are increased by one only if both the number of nodal parameters and the parameter m are odd numbers. This result is illustrated in Fig. 4.

Natural is the question if the derived integration rules are re-discovered, or are they new. And if they are new, how they can be compared with the existing rules.

Let us start from Table 2. For odd m the rules coincide with Gregory formulas [6]. For even m these are new rules, though they are worse in the sense of the degree of precision if compared with Gregory formulas. This can be explained by the property of finite element direct stiffness procedure used to sum the interjacent integrals. For even m the finite elements do not have a central node. Instead, all middle finite elements are integrated over one middle section between 2 closest nodes around $\zeta = 0$ (Eq. (25); compare with Fig. 1). Then the difference $\Delta_i(t)$ (Eq. (46)) becomes an even function and improvement of the degree of precision is impossible. In this sense the proposed derivation methodology has a definite drawback compared with Gregory formulas. Nevertheless the main advantage of the finite element methodology is its universality which reveals its power when the number of nodal parameters is higher.

Table 3 presents a set of rules with 2 nodal parameters. At $m = 2$ the derived rule coincides with Euler–Maclaurin formula of degree 4 [12]. But at $m = 3$ the derived rule does not coincide with Euler–Maclaurin formula of degree 6. Closed form Euler–Maclaurin formulas contain Hermite conditions only at the first and the last node of the equally spaced mesh. Rules with 3 nodal parameters are presented in Table 4. At $m = 2$ the derived rule coincides with Euler–Maclaurin formula of degree 6. Though there exist numerous variations of Hermite type integration rules (closed, half closed, open form), we could not find existing rules for higher m and higher number of nodal parameters. Lots of efforts have been spent for adapting Hermite type integration rules for functions containing singularities [6]. Then adaptive step partition manipulations together with Hermite conditions can produce excellent results [13,14], but that is out of scope of interest in this paper. It could be also mentioned that the derived compound integration rules can be modified by repeated elimination of the leading contribution to the error what would lead to Romberg type quadrature rules [1,3]. Again, this is out of scope of interest in this paper.

The proposed derivation procedure of closed symmetric integration rules with equally spaced steps can look rather complicated and requiring calculation of a lot of definite integrals. It can be noted that all calculations are presented

Table 4
Nodal weights of the integration rule with nodal values of f_i, f_i' and f_i''

m	2	3	4	5	6	7
a_0	1	1	1	1	1	1
a_1	$\frac{1}{2}$	$\frac{468\,627}{1\,146\,880}$	$\frac{7031}{18\,711}$	$\frac{17\,305\,794\,401\,515}{48\,971\,284\,217\,856}$	$\frac{417\,106\,011\,693}{1\,244\,672\,000\,000}$	$\frac{48\,180\,824\,039\,771\,567\,965\,037}{150\,181\,475\,046\,653\,952\,000\,000}$
a_2	1	$\frac{233}{210}$	$\frac{38\,501}{29\,568}$	$\frac{204\,438\,506\,933}{125\,954\,949\,120}$	$\frac{20\,345\,846\,219}{9\,073\,658\,880}$	$\frac{206\,011\,279\,954\,107\,588\,811}{64\,313\,008\,137\,625\,927\,680}$
a_3		$\frac{3\,378\,247}{3\,440\,640}$	$\frac{373}{462}$	$\frac{643}{2310}$	$-\frac{15\,202\,822}{8\,860\,995}$	$-\frac{3\,158\,402\,637\,978\,950\,168\,951}{397\,571\,323\,032\,596\,643\,840}$
a_4			$\frac{2\,430\,347}{2\,395\,008}$	$\frac{38\,377\,978\,002\,737}{30\,607\,052\,636\,160}$	$\frac{37\,276\,477}{12\,446\,720}$	$\frac{282\,689}{22\,610}$
a_5				$\frac{332\,943\,261\,457}{335\,879\,864\,320}$	$\frac{184\,995\,864\,527}{290\,357\,084\,160}$	$-\frac{1\,621\,678\,895\,300\,798\,844\,553}{397\,571\,323\,032\,596\,643\,840}$
a_6					$\frac{222\,914\,844\,343}{221\,524\,875\,000}$	$\frac{5\,128\,938\,983\,828\,079\,463\,338\,377}{3\,416\,628\,557\,311\,377\,408\,000\,000}$
a_7						$\frac{66\,953\,476\,438\,970\,650\,141}{67\,281\,300\,820\,900\,970\,496}$
$\sum_{i=1}^m a_i$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$	$\frac{9}{2}$	$\frac{11}{2}$	$\frac{13}{2}$
p	5	9	11	15	17	21
b_0	0	0	0	0	0	0
b_1	$\frac{1}{10}$	$\frac{72\,567}{1\,146\,880}$	$\frac{544}{10\,395}$	$\frac{745\,044\,996\,725}{16\,323\,761\,405\,952}$	$\frac{35\,402\,551\,029}{871\,270\,400\,000}$	$\frac{267\,306\,874\,875\,933\,237\,877}{7\,230\,959\,909\,653\,708\,800\,000}$
b_2	0	$-\frac{4619}{143\,360}$	$\frac{2987}{114\,048}$	$\frac{59\,706\,290\,989}{412\,216\,197\,120}$	$\frac{564\,486\,975\,959}{1\,512\,276\,480\,000}$	$\frac{5\,403\,390\,123\,590\,504\,612\,107}{7\,592\,507\,905\,136\,394\,240\,000}$
b_3		$\frac{7031}{1\,146\,880}$	$\frac{70\,423}{997\,920}$	$\frac{471\,049\,895\,711}{4\,534\,378\,168\,320}$	$-\frac{2\,681\,731\,110\,787}{10\,585\,935\,360\,000}$	$-\frac{64\,125\,927\,127\,057\,863\,413\,447}{30\,370\,031\,620\,545\,576\,960\,000}$
b_4			$-\frac{5941}{1\,330\,560}$	$-\frac{549\,606\,126\,943}{5\,829\,914\,787\,840}$	$-\frac{50\,324\,291\,011}{90\,478\,080\,000}$	$-\frac{977\,475\,665\,283\,693\,302\,471}{949\,063\,488\,142\,049\,280\,000}$
b_5				$\frac{116\,707\,797}{47\,982\,837\,760}$	$\frac{5\,479\,922\,436\,227}{42\,343\,741\,440\,000}$	$\frac{4\,655\,898\,921\,426\,303\,093\,323}{2\,760\,911\,965\,504\,143\,360\,000}$
b_6					$-\frac{337\,296\,347}{206\,756\,550\,000}$	$-\frac{586\,824\,618\,897\,907\,962\,107}{3\,451\,139\,956\,880\,179\,200\,000}$
b_7						$\frac{19\,424\,780\,649\,112\,775}{16\,197\,350\,197\,624\,307\,712}$
$\sum_{i=1}^m b_i$	$\frac{1}{10}$	$\frac{21\,323}{573\,440}$	$\frac{57\,731}{399\,168}$	$\frac{118\,071\,449\,341}{582\,991\,478\,784}$	$-\frac{4\,050\,588\,994\,669}{15\,122\,764\,800\,000}$	$-\frac{527\,413\,612\,979\,965\,885\,553}{602\,579\,992\,471\,142\,400\,000}$
p	5	9	11	15	17	21
c_0	$\frac{1}{60}$	$\frac{1943}{71\,680}$	$\frac{745}{33\,264}$	$\frac{277\,671\,235}{10\,796\,138\,496}$	$\frac{67\,959\,973}{2\,800\,512\,000}$	$\frac{172\,429\,570\,836\,366\,193}{6\,779\,024\,915\,300\,352\,000}$
c_1	$\frac{1}{20}$	$\frac{4329}{1\,146\,880}$	$\frac{17}{6237}$	$\frac{17\,681\,479\,625}{8\,161\,880\,702\,976}$	$\frac{155\,878\,227}{87\,127\,040\,000}$	$\frac{9\,980\,849\,704\,717\,234\,819}{6\,507\,863\,918\,688\,337\,920\,000}$
c_2	$\frac{1}{60}$	$\frac{10\,051}{258\,048}$	$\frac{16\,249}{266\,112}$	$\frac{9\,500\,590\,123}{107\,961\,384\,960}$	$\frac{133\,188\,945\,397}{1\,058\,593\,536\,000}$	$\frac{30\,130\,767\,945\,731\,493\,823}{175\,211\,720\,887\,762\,944\,000}$
c_3		$\frac{273\,599}{10\,321\,920}$	$\frac{1901}{332\,640}$	$-\frac{1\,209\,995\,473}{19\,377\,684\,480}$	$-\frac{318\,112\,621\,103}{1\,058\,593\,536\,000}$	$-\frac{8\,177\,517\,189\,123\,713\,434\,979}{9\,111\,009\,486\,163\,673\,088\,000}$
c_4			$\frac{90\,913}{3\,991\,680}$	$\frac{859\,490\,879\,719}{20\,404\,701\,757\,440}$	$\frac{8\,875\,460\,311}{39\,207\,168\,000}$	$\frac{131\,565\,767\,224\,689\,072\,151}{94\,906\,348\,814\,204\,928\,000}$
c_5				$\frac{38\,592\,183\,053}{1\,511\,459\,389\,440}$	$\frac{24\,974\,306\,963}{4\,234\,374\,144\,000}$	$-\frac{3\,562\,763\,582\,204\,547\,656\,029}{9\,111\,009\,486\,163\,673\,088\,000}$
c_6					$\frac{64\,527\,750\,293}{2\,646\,483\,840\,000}$	$\frac{527\,853\,270\,550\,774\,109\,713}{11\,388\,761\,857\,704\,591\,360\,000}$
c_7						$\frac{231\,019\,546\,137\,184\,022\,767}{9\,111\,009\,486\,163\,673\,088\,000}$
$\sum_{i=1}^m c_i$	$\frac{1}{40}$	$\frac{1985}{28\,672}$	$\frac{877}{9504}$	$\frac{686\,481\,605}{7\,197\,425\,664}$	$\frac{42\,646\,633}{509\,184\,000}$	$\frac{357\,613\,114\,421\,283\,763}{1\,042\,926\,910\,046\,208\,000}$
p	5	9	11	15	17	21
l	6	10	12	16	18	22

explicitly only for clarity. In fact finite element techniques are exploited to derive the coefficients of the symmetric quadrature rules. Particularly, symbolic calculations are used as the coefficients of the integration rules are usually presented as rational numbers. Therefore the derivation of the rules is straightforward, universal and rather simple. Shape functions of Lagrange finite elements, direct stiffness procedure, solution of a system of algebraic equations are standard objects in any finite element package. No sophisticated techniques of combinatorial algebra or functional analysis are required for that purpose. Every engineer with basic symbolic programming skills and some knowledge of finite element techniques can easily derive the integration rule of any required degree and specified number of nodal derivatives of the integrand.

Finally it can be noted that the derived quadrature rules can be very effective in such situations when the number of nodes is not known at the beginning of the integration process what is common in experimental analysis where the nodal values of the integrand are generated as sequences of discrete numbers.

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