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Existence of solitary solutions in a class of nonlinear differential equations with polynomial nonlinearity

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ABSTRACT

The inverse balancing method for the determination of the necessary conditions of existence of solitary solutions to mth order differential equations with nth order polynomial nonlinearity is presented in this paper. It is shown that the order of possible solitary solutions does not increase if orders of the differential equation and the polynomial nonlinearity increase. Furthermore, the relationships between the order of the solitary solution and the order of the equation (and the nonlinearity) are given in the explicit form.

1. Introduction

With the growth of computational power, a number of methods based on symbolic computations for the construction of solutions to nonlinear differential equations have been developed during the recent decades. Special attention has been devoted to solitary solutions of differential equations with polynomial nonlinearity. Traveling waves in a one-dimensional model of hemodynamics are studied in [1]; the solitary solutions of the variant Boussinesq equations used in water wave modeling are considered in [2,3]. Four aspects of solitary wave solutions of high-level Green–Naghdi equations are discussed in [4]. Exact solitary solutions to the Kuramoto–Sivashinsky equation, which describes the fluctuation of the position of a flame front, are considered in [5] utilizing the consistent Riccati expansion method and in [6,7] with the tanh method. In [8], the $(\frac{G'}{G})$-expansion method has been applied to study solitary solutions of Fisher's equation, which describes the process of interaction between diffusion and reaction. The same method has been applied to the Klein–Gordon equation arising in quantum field theory [9]. The Exp-function method has been used to compute solitary solutions to the Dullin–Gottwald–Holm equation used in hydrodynamics [10] and the Cahn–Allen equation describing the process of phase separation in iron alloys [11].

The Exp-function method [12,13], the tanh-function method [14,15], the $(\frac{G'}{G})$ expansion method [16,17] are typical examples of techniques for the identification of closed-form solitary solutions to nonlinear evolutions in mathematical physics. However, straightforward application of these methods has attracted a considerable amount of criticism [18–20].

One of the main criticisms of the Exp-function method is that the obtained solitary solutions do not always satisfy the differential equation in the general case [18,20].

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A class of simplest equation methods for the determination of exact solutions to nonlinear differential equations, first introduced by Kudryashov in [21], does not possess the drawbacks of the Exp-function method. The basis of the simplest equation method is to use the solutions of the simplest nonlinear differential equations to express the solution of the given equation [21]. The simplest equation method has been extended and applied to the Sharma–Tasso–Olver and Burgers–Huxley equations in [22]. The exact solutions of a model describing the pattern formation processes on the semiconductor surfaces under ion beam bombardment are considered using the simplest equation method in [23]. The method has been generalized for application to non-autonomous differential equations and applied to the Painlevé equations in [24].

The modification of the Simplest equation method, due to Vitanov et al. in [25–27] is used to obtain exact traveling-wave solutions for two classes of model PDEs from ecology and population dynamics [28]. The modified simplest equation method has been demonstrated to yield solitary wave solutions for nonlinear partial differential equations in [29] and has been applied to compute traveling-wave solutions to the Swift–Hohenberg and generalized Rayleigh equations [26] as well as the generalized Kuramoto–Sivashinsky, reaction–diffusion equation with density-dependent diffusion, and the reaction-telegraph equations [25].

The main objective of this paper is to demonstrate an analytical framework based on the simplest equation method for the identification of solitary solutions to the following partial differential equation:

$$\frac{\partial^m u}{\partial t^m} + A_{m-1,0}\frac{\partial^{m-1} u}{\partial t^{m-1}} + A_{0,m-1}\frac{\partial^{m-1} u}{\partial z^{m-1}} + \cdots + A_{10}\frac{\partial u}{\partial t} + A_{01}\frac{\partial u}{\partial z} = a_n u^n + \cdots + a_0, \tag{1}$$

where $A_{j,r}, a_k \in \mathbb{R}$; $j, r = 1, \ldots, m-1$, $k = 0, \ldots, n$ and $a_n \neq 0$. The wave variable substitution $x := kt + \omega z$; $k, \omega \in \mathbb{R}$ transforms (1) to the m-th order differential equation with constant coefficients and nth order polynomial nonlinearity:

$$y_x^{(m)} + b_{m-1}y_x^{(m-1)} + \cdots + b_1 y'_x = a_n y^n + a_{n-1}y^{n-1} + \cdots + a_0, \tag{2}$$

with $b_j \in \mathbb{R}$, $j = 1, \ldots, m$.

The necessary conditions of existence of the solitary solution to (1):

$$y_0 = y_0(x) = \sigma\frac{\prod_{j=1}^{l}(e^{\eta(x-c)} - y_j)}{\prod_{j=1}^{l}(e^{\eta(x-c)} - x_j)}, \tag{3}$$

where $l \in \mathbb{N}, \sigma, \eta, c \in \mathbb{R}, \sigma, \eta \neq 0; y_j, x_j \in \mathbb{C}, j = 1, \ldots, l$ are derived in terms of the equation order n, m and the solution order l.

Nonlinear partial differential equations of the form (1) have already been considered in literature. Nonlinear equations with polynomial nonlinearity up to the fourth order that admit solitary solutions are discussed in [30,31]. A discussion of the existence of exact solutions for a seventh order nonlinear partial differential equation can be found in [32]. The paper [33] contains an extensive discussion on the solitary solutions of various well-known nonlinear partial differential equations that include the form of the solution (3) and variants of Eq. (1). Traveling wave solutions to Eq. (1) without mixed derivatives with nonlinearities up to the fifth order are derived using the modified simplest equation method in [34]. Polynomial nonlinearities in differential equations that model interacting populations are discussed in [35].

Our approach is to determine the parameters of the differential equation in terms of the parameters of the solution. This technique allows the determination of constraints on the order of the differential equation, the nonlinearity terms and the solitary solution for Eq. (1) to admit solitary solutions (3). It is also demonstrated that if the condition on n, m and l is satisfied, additional constraints on the parameters of (2) and (3) must be imposed to ensure the existence of (3) as a solution to (1).

2. Inverse balancing method

2.1. Simplification of (3)

The variable substitution $\hat{x} := e^{\eta(x-c)}$ is introduced. Then, (3) reads:

$$y_0(x) = \hat{y}_0(\hat{x}) = \sigma\frac{Y_l(\hat{x})}{X_l(\hat{x})}, \quad Y_l(\hat{x}) := \prod_{j=1}^{l}(\hat{x} - y_j), \quad X_l(\hat{x}) := \prod_{j=1}^{l}(\hat{x} - x_j). \tag{4}$$

Note that

$$y'_x = \eta\hat{x}\hat{y}'_{\hat{x}}, \quad y_x^{(k)} = (\eta\hat{x}\hat{y}'_{\hat{x}})_{\hat{x}}^{(k-1)} = \eta^k\sum_{j=1}^{k}c_{kj}\hat{y}_{\hat{x}}^{(j)}\hat{x}^j, \quad k = 2, 3, \ldots, \tag{5}$$

where $c_{kj} \in \mathbb{R}$, $j = 1, \ldots, k$,. Then (2) reads:

$$\eta^m\hat{x}^m\hat{y}_{\hat{x}}^{(m)} + \hat{b}_{m-1}\hat{x}^{m-1}\hat{y}_{\hat{x}}^{(m-1)} + \cdots + \hat{b}_1\hat{x}\hat{y}'_{\hat{x}} = a_n\hat{y}^n + a_{n-1}\hat{y}^{n-1} + \cdots + a_0. \tag{6}$$

The coefficients $\hat{b}_k$, $k = 1, \ldots, m-1$ are linear combinations of c_{kj}.

Table 1
Table of necessary existence conditions of solitary solutions to (1). ∃ denotes existence with all parameter values, ∃* denotes existence with additional constraints on parameters, ∄ denotes the nonexistence of solitary solutions.

l	(n, m)						
	(2,1)	(3,2)	(4,3)	(5,4)	(6,5)	(7,6)	(8,7)
1	∃	∃*	∃*	∃*	∃*	∃*	∃*
2	∄	∃*	∃*	∃*	∃*	∃*	∃*
3	∄	∃*	∃*	∃*	∃*	∄	∄
4	∄	∄	∄	∄	∄	∄	∄
5	∄	∄	∄	∄	∄	∄	∄

2.2. Balancing of n and m

Inserting (4) into (6) yields:

$$\left(\frac{\widetilde{Y}_{lm}(\widehat{x})}{X_l^{m+1}(\widehat{x})}+\cdots+\frac{\widehat{b}_1}{\eta^m}\frac{\widetilde{Y}_{l1}(\widehat{x})}{X_l^2(\widehat{x})}\right)=\frac{1}{\sigma\eta^m}\left(a_0+a_1\frac{Y_l(\widehat{x})}{X_l(\widehat{x})}+\cdots+a_{m-1}\frac{Y_l^n(\widehat{x})}{X_l^n(\widehat{x})}\right), \tag{7}$$

where $\widetilde{Y}_{lk}, k=1,\ldots,m$ are polynomials in $\widehat{x}$ of degree $(k+1)l-1$. It can be seen that the equality can hold only when the highest derivative and nonlinear term is balanced:

$$n=m+1. \tag{8}$$

This condition has been noted in literature [20,36]. The relation (8) can also be obtained as the balance equation of the modified simplest equation method [25,26].

2.3. Inverse balancing of the coefficients of the differential equation

Suppose that $n=m+1$, the parameters of (4) are fixed and the parameters of (6) are unknown. Then (7) results in the polynomial equality:

$$\left(\widetilde{Y}_{lm}(\widehat{x})+\frac{\widehat{b}_{m-1}}{\eta^m}\widetilde{Y}_{l,m-1}(\widehat{x})X_l(\widehat{x})+\cdots+\frac{\widehat{b}_1}{\eta^m}\widetilde{Y}_{l1}(\widehat{x})X_l^{m-1}(\widehat{x})\right)$$
$$=\frac{1}{\eta^m}\left(a_0X_l^{m+1}(\widehat{x})+a_1X_l^m(\widehat{x})Y_l(\widehat{x})+\cdots+a_{m+1}Y_l^{m+1}(\widehat{x})\right). \tag{9}$$

Eq. (9) yields a system of linear equations of order $q_{lm}=(m+1)l+1$ with respect to $\widehat{b}_j, a_k, j=1,\ldots,m; k=0,\ldots,m+1$. Because the obtained linear system has more equations than unknowns, additional conditions are required to prove that it is consistent.

Note that for given m and l the number of parameters in (3) is $v_l=2l+2$; the number of parameters in (2) is $u_m=2m+1$, thus the total number of parameters is $p_{lm}=v_l+u_m$. However, the dimension of the obtained linear system is q_{lm}, thus (3) can be a solution to (2) if the following necessary condition holds:

$$q_{lm}\le p_{lm}. \tag{10}$$

Note that the dimension q_{lm} increases nonlinearly in l, m, thus there is a limited number of cases for which (10) holds. The cases of n, m and l for which solitary solutions can exist are listed in Table 1.

3. Examples

3.1. Riccati equation

Let $l=m=1$ in (3) and (2). The resulting equation is the Riccati equation [37]. Note that $q_{11}=3$, $p_{11}=6$ and, by (8), $n=2$. It can be verified that (10) holds. Then

$$y_0(x)=\sigma\frac{e^{\eta(x-c)}-y_1}{e^{\eta(x-c)}-x_1};\quad \widehat{y}_0(\widehat{x})=\sigma\frac{\widehat{x}-y_1}{\widehat{x}-x_1}. \tag{11}$$

Eq. (9) reads:

$$\eta\sigma\widehat{x}(y_1-x_1)=a_0(\widehat{x}-x_1)^2+a_1\sigma(\widehat{x}-y_1)(\widehat{x}-x_1)+a_2\sigma^2(\widehat{x}-y_1)^2. \tag{12}$$

Taking $\widehat{x}=x_1$, $\widehat{x}=y_1$, $\widehat{x}=0$ in (12) yields a system of equations:

(i) $a_2\sigma(y_1-x_1)=\eta x_1$;

(ii) $a_0(y_1 - x_1) = \eta y_1 \sigma$;
(iii) $a_0 x_1^2 + a_1 \sigma x_1 y_1 + a_2 \sigma^2 y_1^2 = 0.$ (13)

The solution to (13) reads:

$$a_0 = \frac{\eta \sigma y_1}{y_1 - x_1}, \quad a_1 = \frac{\eta(x_1 + y_1)}{x_1 - y_1}, \quad a_2 = \frac{\eta x_1}{\sigma(y_1 - x_1)}. \tag{14}$$

Note that (14) and subsequent expressions of the differential equation coefficients hold in the nondegenerate case, when the denominators of the respective expressions are nonzero.

Since $q_{11} = u_1$, there are no additional constraints imposed on σ, x_1, y_1, thus the inverse balancing method describes the following mapping:

$$(\eta, \sigma, y_1, x_1) \to (a_0, a_1, a_2). \tag{15}$$

This means that all solutions of the Riccati equation are of the form (11). This has been reported in literature [37,38].

3.2. Huxley equation

Suppose that $m = 2$. The resulting differential equation is the Huxley equation [37]:

$$y''_x + b_1 y'_x = a_3 y^3 + a_2 y^2 + a_1 y + a_0. \tag{16}$$

After the variable substitution (16) reads:

$$\eta^2 \widehat{x}^2 \widehat{y}''_{\widehat{x}} + (\eta^2 + b_1 \eta)\widehat{x}\widehat{y}'_{\widehat{x}} = a_3 \widehat{y}^3 + a_2 \widehat{y}^2 + a_1 \widehat{y} + a_0. \tag{17}$$

3.2.1. First order solitary solution

Let $l = 1$, then $q_{12} = 4$, $p_{12} = 9$, $n = 2$, thus (10) is satisfied. It has been shown in [39] that the Riccati equation can be extended to the Huxley equation. Thus, (16) has first order solitary solutions (11) if the equality

$$\left(b_1 - 2\frac{\widehat{a}_2 a_2}{a_3}\right)\left(\left(b_1 + \frac{\widehat{a}_2 a_2}{a_3}\right) + 9\frac{\widehat{a}_2^2 a_1}{a_3}\right)\left(b_1 + \frac{\widehat{a}_2 a_2}{a_3}\right) = 27\frac{\widehat{a}_2^3 a_0}{a_3} \tag{18}$$

is satisfied and the initial conditions $y|_{x=c} = s, y'_t|_{x=c} = t$ lie on the curve

$$t = \widehat{a}_0 + \widehat{a}_1 \tilde{\ } s + \widehat{a}_2 s^2, \tag{19}$$

where

$$y'_x = \widehat{a}_0 + \widehat{a}_1 y + \widehat{a}_2 y^2; \quad \widehat{a}_k \in \mathbb{R},\ k = 1, 2, 3; \tag{20}$$

is the Riccati (narrowed) equation [39].

3.2.2. Second order solitary solution

Let $l = 2$, then $q_{22} = 7$, $p_{22} = 10$, $n = 3$ and (10) holds. Substituting (4) with $l = 2$ into (17) and simplifying yields:

$$\begin{aligned}
&\eta\sigma\widehat{x}((X_2(\widehat{x}))^2(2\widehat{x} - y_2 - y_1) - X_2(\widehat{x})Y_2(\widehat{x})(2\widehat{x} - x_2 - x_1))b_1 \\
&\quad + \eta\sigma\widehat{x}(2(X_2(\widehat{x}))^2\eta\widehat{x} - 2X_2(\widehat{x})Y_2(\widehat{x})\eta\widehat{x} - 2X_2(\widehat{x})(2\widehat{x} - y_2 - y_1) \times (2\widehat{x} - x_2 - x_1)\eta\widehat{x} + 2Y_2(\widehat{x})(2\widehat{x} - x_2 - x_1)^2\eta\widehat{x}) \\
&= a_3\sigma^3(Y_2(\widehat{x}))^3 + a_2\sigma^2(Y_2(\widehat{x}))^2 X_2(\widehat{x}) + a_1\sigma Y_2(\widehat{x})(X_2(\widehat{x}))^2 + a_0(X_2(\widehat{x}))^3.
\end{aligned}$$

Taking $\widehat{x} = x_1$, $\widehat{x} = x_2$, $\widehat{x} = y_1$, $\widehat{x} = y_2$, $\widehat{x} = (x_1 + x_2)/2, \widehat{x} = (y_1 + y_2)/2$ and $\widehat{x} = 0$ results in $q_{22} = 7$ linear equations. The solution of this system of equations is given in Appendix A. The linear system is consistent when the following conditions are satisfied:

(i) $\dfrac{Y_2(x_2)}{Y_2(x_1)} = \dfrac{x_2}{x_1}$,
(ii) $\sigma^3 {y_1}^3 {y_2}^3 a_3 + \sigma^2 x_1 x_2 {y_1}^2 {y_2}^2 a_2 + \sigma {x_1}^2 {x_2}^2 y_1 y_2 a_1 + {x_1}^3 {x_2}^3 a_0 = 0.$ (21)

Note that from (21), the parameters y_1, y_2 can be expressed in terms of the remaining parameters: $y_1 = y_1(\sigma, x_1, x_2), y_2 = y_2(\sigma, x_1, x_2)$. Explicit expressions of y_1 and y_2 are omitted for brevity.

Thus, the inverse balancing method in this case describes the mapping:

$$(\eta, \sigma, x_1, x_2) \to (a_0, a_1, a_2, a_3, b_1). \tag{22}$$

4. Concluding remarks

It is shown that the inverse balancing methods yields the necessary conditions of existence for solitary solutions to the mth order differential equation with constant coefficients and polynomial nonlinearity. A computational framework for the determination of these conditions is presented. It is demonstrated that as the order of the equation and the nonlinearity increases, new solitary solutions of higher order do not appear. Inverse balancing yields that there can only be solitary solutions of order $l = 1, 2, 3$ in the general case and additional constraints on the parameters of the solution and the equation must be imposed in order to ensure existence.

It is clear that using the method provides a solid foundation for the application of direct solution construction methods, because it narrows the possible solitary solution set and provides relationships between the system and solution parameters which cannot be obtained directly.

Application of the inverse balancing method to other nonlinear differential equations and systems of equations, as well as using a different form of solution is a definite object of future research.

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Appendix A. Solution of the linear system in 3.2.2

If (21) is satisfied, the system has a unique solution, which reads:

$$a_3 = 2\left(\frac{\eta x_1(x_1 - x_2)}{\sigma Y_2(x_1)}\right)^2; \tag{23}$$

$$a_0 = -\frac{2y_1y_2\sigma\eta^2(\Theta_1 + \Theta_2)}{\Omega_1}, \quad b_1 = -\eta + \frac{2(y_2X_2(y_1)\Theta_2 - y_1X_2(y_2)\Theta_1)}{(y_2 - y_1)\Omega_1}, \tag{24}$$

where

$$\begin{aligned}
\Theta_1 &:= y_1X_2(y_2)((y_2 - y_1)(x_1 + x_2 - 2y_2) - X_2(y_1));\\
\Theta_2 &:= y_2X_2(y_1)((y_1 - y_2)(x_1 + x_2 - 2y_1) - X_2(y_2));\\
\Omega_1 &:= X_2(y_1)X_2(y_2)(y_2X_2(y_1) + y_1X_2(y_2)).
\end{aligned} \tag{25}$$

$$\begin{aligned}
a_1 = &\frac{1}{\Omega_2}(\Lambda_{10}a_0 + \Lambda_{13}a_3 + \Lambda_3(2\tau_xX_2(\tau_x)Y_2(\tau_y) - 2\tau_yY_2^2(\tau_x))(\eta^2 + \eta b_1) + \sigma\eta^2X_2(\tau_x)\\
&\times (Y_2(\tau_y)\Phi_1 - Y_2(\tau_x)(\Phi_2 + 2\tau_x^2X_2(\tau_y)Y_2^2(\tau_y) + 2\tau_y^2Y_2(\tau_x)X_2^2(\tau_y))));\\
a_2 = &\frac{1}{\sigma\Omega_2}(-\Lambda_{20}a_0 - \Lambda_{23}a_3 - \Lambda_3X_2(\tau_x)(2\tau_xX_2(\tau_y) - 2\tau_yY_2(\tau_x))(\eta^2 + \eta b_1) + \sigma\eta^2X_2(\tau_x)\\
&\times (Y_2(\tau_x)X_2(\tau_y)\Phi_1 + X_2^2(\tau_x)(\Phi_2 + 2\tau_y^2X_2^2(\tau_y)Y_2(\tau_x) - 2\tau_x^2X_2^2(\tau_y)Y_2(\tau_y)))),
\end{aligned} \tag{26}$$

where

$$\begin{aligned}
&\tau_x := \frac{x_1 + x_2}{2}, \quad \tau_y := \frac{y_1 + y_2}{2}, \quad \Phi_1 := 2\tau_x^2X_2(\tau_y)X_2(\tau_x)Y_2(\tau_y);\\
&\Phi_2 := 2\tau_y^2Y_2(\tau_x)Y_2(\tau_y)\big(4(\tau_x - \tau_y)^2 - X_2(\tau_y)\big);\\
&\Lambda_{10} := X_2(\tau_x)Y_2^2(\tau_x)X_2^3(\tau_y) - X_2(\tau_y)Y_2^2(\tau_y)X_2^3(\tau_x);\\
&\Lambda_{13} := \sigma^3\big(X_2(\tau_x)Y_2^2(\tau_x)Y_2^3(\tau_y) - X_2(\tau_y)Y_2^2(\tau_y)Y_2^3(\tau_x)\big);\\
&\Lambda_{20} := Y_2(\tau_y)X_2^2(\tau_y)X_2^3(\tau_x) - Y_2(\tau_x)X_2^2(\tau_y)X_2^3(\tau_y);\\
&\Lambda_{23} := \sigma^3\big(Y_2(\tau_y)X_2^2(\tau_y)Y_2^3(\tau_x) - Y_2(\tau_y)X_2^2(\tau_x)Y_2^3(\tau_y)\big);\\
&\Lambda_3 := 2\sigma\eta X_2(\tau_y)X_2(\tau_x)Y_2(\tau_y)(\tau_x - \tau_y);\\
&\Omega_2 := \sigma X_2(\tau_x)X_2(\tau_y)Y_2(\tau_x)Y_2(\tau_y)(Y_2(\tau_y)X_2(\tau_x) - Y_2(\tau_x)X_2(\tau_y)).
\end{aligned}$$

References

[1] A.M. Barlukova, A.A. Cherevko, A.P. Chupakhin, Traveling waves in a one-dimensional model of hemodynamics, J. Appl. Mech. Tech. Phys. 55 (2014) 917–926.
[2] W. Yuana, F. Menga, Y. Huangc, Y. Wud, All traveling wave exact solutions of the variant Boussinesq equations, Appl. Math. Comput. 268 (2015) 865–872.
[3] R. Abazari, The (G'/G)-expansion method for the coupled Boussinesq equation, Procedia Eng. 10 (2011) 2845–2850.
[4] B. Zhao, R. Ertekin, W. Duan, M. Hayatdavoodi, On the steady solitary-wave solution of the Green–Naghdi equations of different levels, Wave Motion 51 (2014) 1382–1395.

[5] M. Chen, H. Hu, H. Zhu, Consistent Riccati expansion and exact solutions of the Kuramoto–Sivashinsky equation, Appl. Math. Lett. 49 (2015) 147–151.
[6] S. Sahoo, S.S. Ray, New approach to find exact solutions of time-fractional Kuramoto–Sivashinsky equation, Physica A 434 (2015) 240–245.
[7] L. Wazzan, A modified tanhcoth method for solving the general Burgers–Fisher and the Kuramoto–Sivashinsky equations, Commun. Nonlinear Sci. Numer. Simul. 14 (2009) 2642–2652.
[8] W.M. Taha, M.S.M. Noorani, Application of the G'/G-expansion method for the generalized Fishers equation and modified equal width equation, J. Assoc. Arab Univ. Basic Appl. Sci. 15 (2014) 82–89.
[9] M.G. Hafez, M.N. Alam, M.A. Akbar, Exact traveling wave solutions to the Klein–Gordon equation using the novel (G'/G)-expansion method, Results Phys. 4 (2014) 177–184.
[10] X. Guang-Can, X. Da-Quang, L. Xi-Qiang, Application of Exp-function method to Dullin–Gottwald–Holm equation, Appl. Math. Comput. 210 (2009) 536–541.
[11] A. Bekir, Multisoliton solutions to Cahn–Allen equation using double exp-function method, Phys. Wave Phenom. 20 (2012) 118–121.
[12] J.H. He, X.H. Wu, Exp function method for nonlinear wave equations, Chaos Solitons Fractals 30 (2006) 700–708.
[13] C. Chun, New solitary wave solutions to nonlinear evolution equations by the Exp-function method, Comput. Math. Appl. 61 (2010) 2107–2110.
[14] E. Yusufoglu, A. Bekir, The tanh and the sinecosine methods for exact solutions of the MBBM and the Vakhnenko equations, Chaos Soliton Fractals 38 (2008) 1126–1133.
[15] E.M.E. Zayed, M.A.M. Abdelaziz, Exact solutions for the nonlinear Schrdinger equation with variable coefficients using the generalized extended tanh–function, the sinecosine and the exp-function methods, Appl. Math. Comput. 218 (2011) 2259–2268.
[16] M. Wang, X. Li, J. Zhang, The G'/G expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics, Phys. Lett. A 372 (2008) 417–423.
[17] M.G. Hafez, M.N. Alam, M.A. Akbar, Exact traveling wave solutions to the Klein–Gordon equationusing the novel G'/G-expansion method, Results Phys. 4 (2014) 177–184.
[18] N.A. Kudryashov, N.B. Loguinova, Be careful with Exp-function method, Commun. Nonlinear Sci. Numer. Simul. 14 (2009) 1891–1900.
[19] Z. Navickas, M. Ragulskis, How far can one go with the Exp-function method? Appl. Math. Comput. 211 (2009) 522–530.
[20] I. Aslan, V. Marinakis, Some remarks on Exp-function method and its applications, Commun. Theor. Phys. 56 (2011) 397–403.
[21] N.A. Kudryashov, Simplest equation method to look for exact solutions of nonlinear differential equations, Chaos Soliton Fractals 24 (2005) 1217–1231.
[22] N.A. Kudryashov, N.B. Loguinova, Extended simplest equation method for nonlinear differential equations, Appl. Math. Comput. 205 (2008) 396–402.
[23] N.A. Kudryashov, P.N. Ryabov, Exact solutions of one pattern formation model, Appl. Math. Comput. 232 (2014) 1090–1093.
[24] A.O. Antonova, N.A. Kudryashov, Generalization of the simplest equation method for nonlinear non-autonomous differential equations, Commun. Nonlinear Sci. Numer. Simul. 19 (2014) 4037–4041.
[25] N.K. Vitanov, Z.I. Dimitrova, H. Kantz, Modified method of simplest equation and its application to nonlinear PDEs, Appl. Math. Comput. 216 (2010) 2587–2595.
[26] N.K. Vitanov, Modified method of simplest equation: Powerful tool for obtaining exact and approximate traveling-wave solutions of nonlinear PDEs, Commun. Nonlinear Sci. Numer. Simul. 16 (2011) 1176–1185.
[27] N.K. Vitanov, On modified method of simplest equation for obtaining exact and approximate solutions of nonlinear PDEs: The role of the simplest equation, Commun. Nonlinear Sci. Numer. Simul. 16 (2011) 4215–4231.
[28] N.K. Vitanov, Z.I. Dimitrova, Application of the method of simplest equation for obtaining exact traveling-wave solutions for two classes of model PDEs from ecology and population dynamics, Commun. Nonlinear Sci. Numer. Simul. 15 (2010) 2836–2845.
[29] N.K. Vitanov, Z.I. Dimitrova, Solitary wave solutions for nonlinear partial differential equations that contain monomials of odd and even grades with respect to participating derivatives, Appl. Math. Comput. 247 (2014) 213–217.
[30] N.A. Kudryashov, Polynomials in logistic function and solitary waves of nonlinear differential equations, Appl. Math. Comput. 219 (2013) 9245–9253.
[31] N.A. Kudryashov, D.I. Sinelshchikov, Nonlinear differential equations of the second, third and fourth order with exact solutions, Appl. Math. Comput. 218 (2012) 10454–10467.
[32] N.A. Kudryashov, One method for finding exact solutions of nonlinear differential equations, Commun. Nonlinear Sci. Numer. Simul. 17 (2012) 2248–2253.
[33] N.A. Kudryashov, Meromorphic solutions of nonlinear ordinary differential equations, Commun. Nonlinear Sci. Numer. Simul. 15 (2010) 2778–2790.
[34] N.K. Vitanov, Application of simplest equations of Bernoulli and Riccati kind for obtaining exact traveling-wave solutions for a class of PDEs with polynomial nonlinearity, Commun. Nonlinear Sci. Numer. Simul. 15 (2010) 2050–2060.
[35] N.K. Vitanov, Z.I. Dimitrova, K.N. Vitanov, Traveling waves and statistical distributions connected to systems of interacting populations, Comput. Math. Appl. 66 (2013) 1666–1684.
[36] A. Ebaid, An improvement on the Exp-function method when balancing the highest order linear and nonlinear terms, J. Math. Anal. Appl. 392 (1) (2012) 1–5.
[37] A.D. Polyanin, V.F. Zaitsev, Handbook of Exact Solutions for Ordinary Differential Equations, Chapman and Hall/CRC, 2003.
[38] Z. Navickas, M. Ragulskis, L. Bikulciene, Be careful with the Exp-function method – additional remarks, Commun. Nonlinear Sci. Numer. Simul. 15 (2010) 3874–3886.
[39] Z. Navickas, M. Ragulskis, L. Bikulciene, Special solutions of Huxley differential equation, Math. Model. Anal. 16 (2011) 248–259.