AN ANALYTICAL SCHEME FOR THE ANALYSIS OF MULTI-HUMP SOLITONS

ZENONAS NAVICKAS, TADAS TELKSNYS, INGA TIMOFEJEVA, MINVYDAS RAGULSKIS

Center for Nonlinear Systems, Kaunas University of Technology, Studentu 50-147
Kaunas LT-51368, Lithuania
tadas.telksnys@ktu.lt

ROMAS MARCINKEVICIUS

Department of Software Engineering, Kaunas University of Technology, Studentu 50-415
Kaunas LT-51368, Lithuania

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An analytical framework for the analysis of multi-hump solitons is proposed in this paper. Multi-hump solitons are defined by imposing special symmetry conditions on the classical soliton expression. Such soliton solutions have a wide range of potential applications in the field of optical communications. The proposed algebras of soliton solutions enable a new look at the propagation dynamics of complex nonlinear wave phenomena. The efficiency of the presented analytical scheme is demonstrated using a system of Riccati differential equations with diffusive and multiplicative coupling.

Keywords: soliton solution; multi-hump soliton; Riccati equation; analytical solution; closed-form solution.

1. Introduction and Motivation

Solitons (or solitary solutions) are nonlinear waves that do not change their shape while they propagate at a constant velocity or after collisions with other solitons. One of the main areas of application for soliton theory is the field of optical communications [7, 3] and, in particular, fiber optics [1]. Seminal experiments conducted by Mollenauer et al demonstrate the viability and power of the application of solitons for error-free transfer of information via optical fibers [14].

More recently, solitons have been successfully used in subpicosecond or femtosecond optical communications. The timing of jitter in a higher-order nonlinear Schrödinger equation with applications in high-speed optical communication systems is discussed in [4]. Yang et al construct one-, two- and three-soliton solutions to a fifth-order nonlinear Schrödinger equation used to model femtosecond pulses in inhomogeneous optical fibers in [25]. A compact nanotube-mode-locked all-fiber laser that can generate picosecond and femtosecond solitons at variable wavelength is presented in [6]. Temporal dissipative solitons that are critical to applications in spectroscopy, astronomy and telecommunications are observed in a continuous-wave laser-driven nonlinear optical microresonator in [8].

In addition to high-speed optical communications, solitons are also an important concept in quantum computing. It is proven in the seminal paper by Knill et al that it is possible to implement general quantum computing using only beam-splitters, phase-shifters, single-photon sources and photo-detectors [10]. In a series of papers [23, 22, 21], Steiglitz has demonstrated that the scheme proposed by Knill et al scheme can be realized using soliton-guided photons.

Traditionally, optical communication schemes use single-hump solitons with a one extremum (maximum and minimum for bright and dark solitons respectively), however, it has been reported that in addition to a stable single-hump soliton, dispersion-managed communication systems can also admit a stable double-hump soliton [19]. Such solitons would enable the use of higher bit-rate communications, since the speed depends on the proximity of single-hump solitons [2].

Double-hump soliton solution to systems of differential equations that have concrete applications in optics have already been constructed. In [12], Liu et al demonstrate that the Sasa-Satsuma equation, which is used to model propagation of ultrashort pulses in a monomode fiber, admits double-hump soliton solutions. The stability of double-hump solitons in a medium with periodic modulation of nonlinear effects is studied in [9]. Dark double-hump soliton solutions to a system of coupled nonlinear Schrödinger equations with third-order dispersion and self-steepening that are used to model pulse propagation in two-mode and birefringent optical fibers are constructed in [24].

The main objective of this paper is to provide a rigorous mathematical framework for the analysis of multi-hump soliton solutions with special symmetry constraints. Even though there are a number of studies that focus on the construction of multi-hump soliton solutions to partial differential equations with applications in high bit-rate optical communications, a complete consideration of the analytical properties of such solutions would help to form a more complete picture of the behavior of such solutions irrespective of the system that is being investigated.

Soliton solutions of the following form are considered:

$$x = \frac{\sum_{j=0}^{N} \alpha_j \exp(j\eta t)}{\sum_{j=0}^{M} \beta_j \exp(j\eta t)}, \tag{1}$$

where $N, M \in \mathbb{N}; N < M; \eta, \alpha_j, \beta_j \in \mathbb{R}; \eta, \alpha_N, \beta_M \neq 0$. While expression (1) is not the only analytical form of soliton solutions, it is inspired by the classical Korteweg-de Vries (KdV) equation [20]:

$$\frac{\partial x}{\partial t} - 6x\frac{\partial x}{\partial \xi} + \frac{\partial^3 x}{\partial \xi^3} = 0, \tag{2}$$

which is widely used in many studies [5, 11, 15]. The KdV equation admits the following soliton solution [20]:

$$x(\xi, t) = -\frac{c}{2}\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(\xi - ct - a)\right), \tag{3}$$

where c and a are the wave propagation speed and arbitrary constant respectively. Note that (3) can be rearranged into the form (1) for $N = 2$, thus (1) is a more general form of KdV-type soliton solutions.

It is clear that expression (1) does not universally result in a physically meaningful soliton solution – for example, it could have up to M singularities and limits that do not decay to zero as $t \to \pm\infty$. Thus we impose the following conditions on solitons with expression (1):

(1) x is an even or odd function;
(2) The limits of x decay to zero as $t \to \pm\infty$;
(3) x has no singularities.

Solitons that satisfy the above conditions are called perfect solitons in the remainder of this paper.

A complete analytical framework for the analysis of solutions (1) with conditions 1–3 is presented in this paper. It is demonstrated that the set of soliton solutions generated by expression (1) includes multi-hump solitons that have potential applications in optical communications. This approach allows enables the analysis of the complex dynamics of nonlinear wave phenomena from a new perspective.

The paper is organized as follows: definitions and basic properties of perfect solitons are given in Section 2; differentiation-related properties of such solutions are discussed in Section 3; perfect soliton solutions to a system of Riccati differential equations with diffusive and multiplicative coupling are considered in Section 4; concluding remarks are given in Section 5.

2. Perfect solitons – properties and relationships

The concept and basic definitions of perfect solitons (solutions with special symmetry properties) are presented in this section.

2.1. *Kink and mother solitons*

Definition 1. The kink soliton is defined as:

$$K(t) := \frac{\exp(t) - 1}{\exp(t) + 1}. \tag{4}$$

Definition 2. The one-parameter family of zero-order perfect even solitons reads:

$$M_a(t) := \frac{(2 + a)\exp(t)}{\exp(2t) + a\exp(t) + 1}; \quad -2 < a < +\infty. \tag{5}$$

In the remainder of the text, functions $M_a(t)$ are called mother solitons – since the term "mother soliton" is taken from wavelet theory, where a mother wavelet function is scaled and shifted to produce wavelet functions. For solitons, just scaling and shifting $M_a(t)$ is not sufficient to produce a full range of solitons that could act as wavelet functions. Sigmoidal kink function is necessary for the generation of odd functions.

The kink is illustrated in Fig. 1 (a); the family of zero-order even solitons is shown in Fig. 1 (b). Solitons $M_a(t)$ defined by (5) are often referred to as bright solitary solutions [20].

Note that the perfect square denominator of the mother soliton obtained in the special case $a = 2$:

$$M_2(t) = \frac{4\exp(t)}{\left(\exp(t)+1\right)^2}. \tag{6}$$

It has the same structural form as the solution to the classical KdV equation [16].

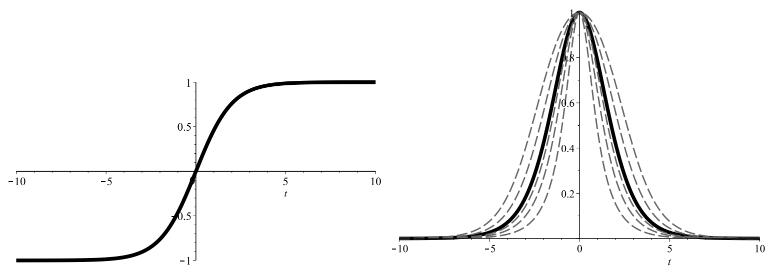

Fig. 1. Kink $K(t)$ (part (a)) and mother solitons $M_a(t)$ (part (b)). The thick black line in (b) corresponds to $M_2(t)$; dashed gray lines above and below $M_2(t)$ correspond to the cases $2 < a < +\infty$ and $-2 < a < 2$ respectively. The parameter a controls the steepness of curve in part (b).

Nonlinear transformations that map any of the functions K, M_2, M_a to the remaining two are given in Fig. 2.

2.2. *Perfect even and odd solitons*

Using kink and mother solitons defined in Subsection 2.1, two classes of perfect solitons can be constructed.

Definition 3. Order $2m, m = 0, 1, \ldots$ perfect even solitons $x_{2m}(t)$ are defined as:

$$x_{2m}\left(t;\sigma,a_0,\ldots,a_m,\beta_1,\ldots,\beta_m\right) := \sigma M_{a_0}(t)\prod_{j=1}^{m}\left(\beta_j + M_{a_j}(t)\right), \tag{7}$$

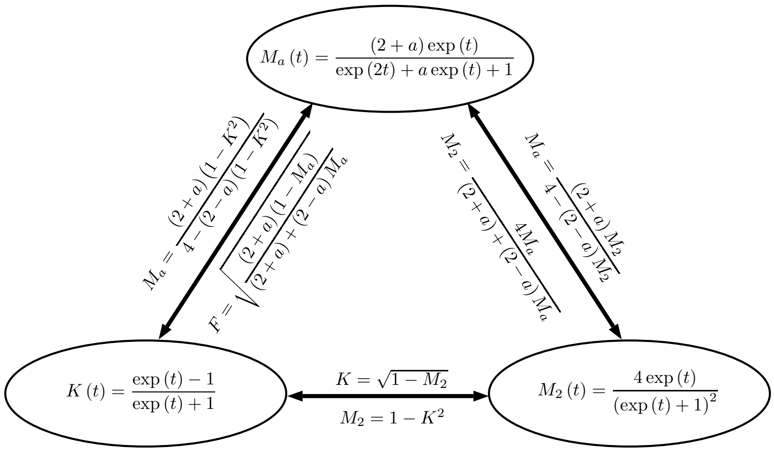

Fig. 2. Transformations between functions M_a, M_2 and K.

where $\sigma \neq 0, \sigma, \beta_j \in \mathbb{R};\ -2 < a_0, a_j < +\infty$. Note that parameters $a_0, \ldots, a_j$ are not necessarily distinct. The set of all perfect even solitons is denoted $\mathbb{S}_e$. The case $m = 0$ results in solitons defined in Definition 2. For brevity, the parameters $\sigma, a_0, \ldots, a_m, \beta_1, \ldots, \beta_m$ will be omitted in the remainder of this paper.

Perfect even solitons have the following properties:

(1) $x_{2m}(-t) = x_{2m}(t)$;
(2) $\lim_{t \to \pm\infty} (x_{2m})_t^{(n)} = 0, \quad n = 0, 1, \ldots$;
(3) x_{2m} has at most $m + 1$ local maxima and minima.

Note that perfect odd solitons are obtained by multiplying perfect even solitons by the sigmoid kink function.

Definition 4. Order $2m+1, m = 0, 1, \ldots$ perfect odd solitons $x_{2m+1}(t)$ are defined as:

$$x_{2m+1}(t; \sigma, a_0, \ldots, a_m, \beta_1, \ldots, \beta_m) := K(t) x_{2m}(t; \sigma, a_0, \ldots, a_m, \beta_1, \ldots, \beta_m), \quad (8)$$

where $x_{2m} \in \mathbb{S}_e$ is any perfect even soliton.

The set of perfect odd solitons defined by (8) is denoted as $\mathbb{S}_o$. Note that $K \notin \mathbb{S}_o$.

Perfect solitons defined by (8) do have the following properties:

(1) $x_{2m+1}(-t) = -x_{2m+1}(t)$;
(2) $\lim_{t \to \pm\infty} (x_{2m+1})_t^{(n)} = 0, \quad n = 0, 1, \ldots$;
(3) x_{2m+1} has at most $m + 2$ local maxima and minima.

Typical examples of perfect even and odd solitons are shown in Fig. 3 (a) and (b) respectively.

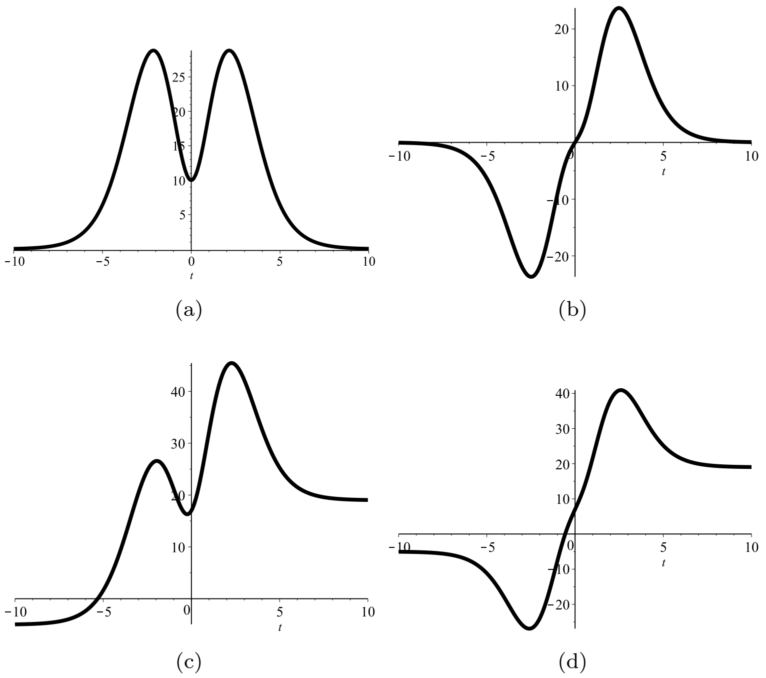

Fig. 3. Perfect even soliton $x_2(t) = -40M_{20}(t)\left(-\frac{5}{4} + M_1(t)\right)$ (part (a)) and perfect odd soliton $x_3(t) = K(t)x_2(t)$ (part (b)). Parts (c) and (d) display perfect odd and even solitons with nonzero limits: $\widetilde{x}_2 = x_2 + 12K(t) + 7$ and $\widetilde{x}_3 = x_3 + 12K(t) + 7$.

The class of perfect solitons can be extended to include solitons with nonzero limits as t tends to infinity. This can be achieved by adding the term $\alpha K(t)+\gamma; \alpha, \gamma \in \mathbb{R}$ to any perfect even soliton:

$$\begin{aligned}\widetilde{x}_{2m}\left(t; \alpha, \gamma, \sigma, a_0, \dots, a_m, \beta_1, \dots, \beta_m\right) &:= x_{2m}\left(t; \sigma, a_0, \dots, a_m, \beta_1, \dots, \beta_m\right) \\ &\quad + \alpha K(t) + \gamma,\end{aligned} \tag{9}$$

where $\widetilde{x}_{2m}$ denotes perfect soliton with the following limits:

$$\lim_{t \to \pm\infty} = \gamma \pm \alpha; \tag{10}$$

and $x_{2m} \in \mathbb{S}_e$. Analogous extensions also hold true for perfect odd solitons x_{2m+1}. The sets of perfect even and odd solitons with nonzero limits read:

$$\widetilde{\mathbb{S}}_e := \mathbb{S}_e \oplus \alpha K(t) \oplus \mathbb{R}; \quad \widetilde{\mathbb{S}}_o := \mathbb{S}_o \oplus \alpha K(t) \oplus \mathbb{R}, \tag{11}$$

where $\oplus$ denotes the direct sum operation.

2.3. *Products of mother solitons*

It can be proven that products of mother solitons can be replaced with sums and vice versa. Such counterintuitive replacements are important in subsequent derivations.

Remark 1. Note that for any $-2 < a, b < +\infty, a \neq b$ the following relation holds true:

$$M_a M_b = \frac{2+b}{b-a} M_a + \frac{2+a}{a-b} M_b. \tag{12}$$

Proof. Let $-2 < a, b < +\infty, a \neq b$. Then:

$$\frac{\exp(2t) + a\exp(t) + 1}{\exp(2t) + b\exp(t) + 1} = 1 + \frac{a-b}{2+b} M_b, \tag{13}$$

thus:

$$\left(1 + \frac{b-a}{2+a} M_a\right)\left(1 + \frac{a-b}{2+b} M_b\right) = 1. \tag{14}$$

Equation (14) results in:

$$\frac{a-b}{2+b} M_b - \frac{a-b}{2+a} M_a = \frac{(a-b)^2}{(2+a)(2+b)} M_a M_b, \tag{15}$$

which yields (12). □

2.4. *Canonical form of perfect solitons*

Perfect even solitons (7) can be rewritten in the expanded form:

$$x_{2m} = \gamma_0 M_{a_0} + \sum_{j=1}^{m} \sum_{\substack{k_1,\dots,k_j \in \{1,\dots,m\} \\ k_1 < \dots < k_j}} \gamma_{k_1,\dots,k_j} M_{a_0} M_{a_{k_1}} \dots M_{a_{k_j}}, \tag{16}$$

where coefficients $\gamma_{k_1,\dots,k_j} \in \mathbb{R}$ are products of β_j.

Note that (12) for any parameters $b_1, \dots, b_n, b_k \neq b_j, k \neq j; n \in \mathbb{N}$ and powers $j_1, \dots, j_n > 0$ results in:

$$M_{b_1}^{j_1} \dots M_{b_n}^{j_n} = \sum_{k=1}^{n} \sum_{s=1}^{j_k} \omega_{ks} M_{b_k}^{s}; \quad \omega_{ks} \in \mathbb{R}. \tag{17}$$

Let $b_1, \dots, b_n; n \leqslant m$ be distinct mother soliton parameters in (7) that occur $j_1, \dots, j_n$ times. The form (16) together with (17) yields that perfect even solitons (7) can be rewritten without mixed products $M_{a_0} M_{a_{k_1}} \dots M_{a_{k_j}}$:

$$x_{2m} = \sum_{k=1}^{n} \sum_{s=1}^{j_k} c_{ks} M_{b_k}^{s}, \tag{18}$$

The canonical form (18) is a unique representation of any perfect even soliton.

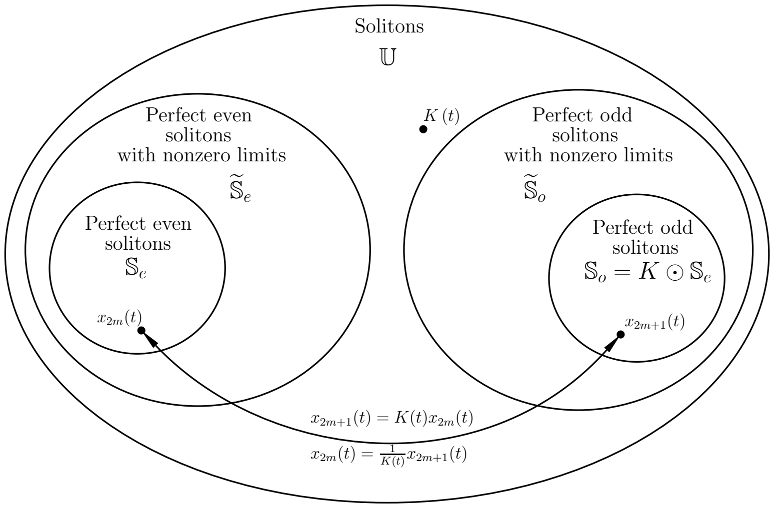

Fig. 4. Subsets of perfect even and odd solitons with zero and nonzero limits within the set of all solitons (set $\mathbb{U}$). Multiplication/division by kink $K(t)$ defines a bijective mapping between $\mathbb{S}_e$ and $\mathbb{S}_o$. The set $\mathbb{S}_e$ forms an Abelian algebra $\mathcal{S}_e$ over $\mathbb{R}$.

For example, let $m = 3, \sigma = 1, a_0 = 2, a_1 = 1, a_2 = a_3 = -1, \beta_1 = 3, \beta_2 = 4, \beta_3 = -2$, then:

$$x_6 = M_2 \left(3 + M_1\right) \left(4 + M_{-1}\right) \left(-2 + M_{-1}\right). \tag{19}$$

Expanding the product in (19) results in:

$$x_6 = -24M_2 + 6M_2M_{-1} - 8M_2M_1 + 3M_2M_{-1}^2 + 2M_2M_1M_{-1} + M_2M_1M_{-1}^2. \tag{20}$$

Using (12) transforms (20) into:

$$x_6 = -35M_1 + 9M_{-1} + 6M_{-1}^2, \tag{21}$$

note that as a result of transformation into the canonical form, the mother soliton M_2 does not appear in (21).

2.5. *Operations on perfect solitons*

Definition 3 and canonical form (18) result in the following properties of sum and product operations on perfect even solitons $x_{2m}, y_{2k}, z_{2l} \in \mathbb{S}_e$:

(1) $\alpha x_{2m} + \beta y_{2k} \in \mathbb{S}_e; \quad \alpha, \beta \in \mathbb{R}$;
(2) $x_{2m} y_{2k} \in \mathbb{S}_e$.

Note that property 1. results directly from (18). The sum operation on two perfect even solitons the form (18) yields a sum of the same form, which uniquely represents a resulting higher-order (compared to x_{2m}, y_{2k}) perfect even soliton.

Remark 2. The set of perfect even solitons $\mathbb{S}_e$ with sum and multiplication operations form an Abelian algebra $\mathcal{S}_e$ over $\mathbb{R}$.

Results on closedness for perfect odd solitons are completely different: the set of perfect odd solitons is closed under the sum operation, but is not closed under multiplication. Let $x_{2m+1}, y_{2n+1} \in \mathbb{S}_o$, then (8) and Fig. 2 yield:

$$x_{2m+1}y_{2n+1} = (Kx_{2m})(Ky_{2n}) = K^2 x_{2m}y_{2n} = (1 - M_2)\, x_{2m}y_{2n} \in \mathbb{S}_e, \tag{22}$$

thus

$$\mathbb{S}_o \odot \mathbb{S}_o \subset \mathbb{S}_e, \tag{23}$$

where $\odot$ denotes the direct set product operation. Furthermore,

$$K \odot \mathbb{S}_e = \mathbb{S}_o; \quad \frac{1}{K} \odot \mathbb{S}_o = \mathbb{S}_e, \tag{24}$$

thus multiplication and division by kink $K(t)$ defines a bijective mapping between the sets $\mathbb{S}_e$ and $\mathbb{S}_o$. A schematic diagram of sets $\mathbb{S}_e, \mathbb{S}_o$ within the set of all solitons $\mathbb{U}$ is given in Fig. 4. Note that for $t = 0$ the ratio is evaluated as:

$$\left.\frac{x_{2m+1}(t)}{K(t)}\right|_{t=0} = \lim_{t \to 0} \frac{x_{2m+1}(t)}{K(t)}. \tag{25}$$

3. Derivatives of perfect solitons

Because the aim of this paper is to consider differential equations that admit perfect soliton solutions, some properties with respect to their derivatives must be considered. In this section it is demonstrated that differentiation of perfect solitons increases their complexity – derivatives of perfect solitons are also perfect solitons, but of a higher order. Furthermore, differentiation maps even solitons to odd and vice versa.

3.1. *Basic properties of $K(t)$ and $M_a(t)$*

It can be observed that the derivatives of $K(t)$ are given in terms of $M_2(t)$:

$$K'_t = \frac{1}{2}M_2; \quad K''_{tt} = -\frac{1}{2}KM_2. \tag{26}$$

Analogous relations can be derived for $M_a(t)$ using transformations given in Fig. 2 and (26):

$$(M_a)'_t = -KM_a\,(1 + \lambda_a M_a)\,; \quad \lambda_a := \frac{2-a}{2+a}. \tag{27}$$

The second derivative of M_a reads:

$$(M_a)''_{tt} = M_a\left(1 + \alpha_a^{(1)} M_a\right)\left(1 + \alpha_a^{(2)} M_a\right), \tag{28}$$

where

$$\alpha_a^{(1,2)} = \frac{8 - 4a}{3a \pm \sqrt{a^2 + 32}}. \tag{29}$$

Proof of (28) is given in Appendix Appendix A.

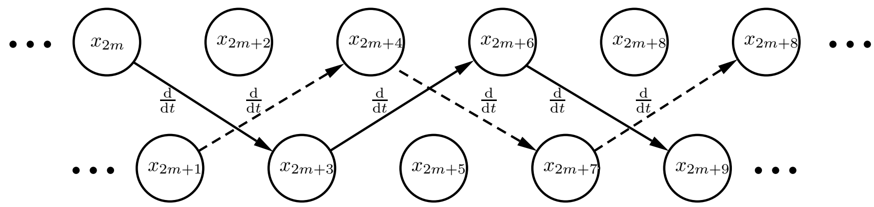

Fig. 5. Schematic illustration of the effect of differentiation on perfect solitons. Note that differentiation maps perfect even solitons to higher-order perfect odd solitons and vice versa. Solid arrows denote differentiation starting at x_{2m}; dashed arrows denote differentiation starting at x_{2m+1}.

3.2. *Derivatives of perfect solitons*

Let $x_{2m} \in \mathbb{S}_e, z_{2k+1} \in \mathbb{S}_o$. Then there exist unique $y_{2(m+1)}, w_{2(k+1)} \in \mathbb{S}_e$ such that:

$$(x_{2m})'_t = K y_{2(m+1)} \in \mathbb{S}_o, \tag{30}$$

and

$$(z_{2k+1})'_t = M_2 w_{2(k+1)} \in \mathbb{S}_e. \tag{31}$$

The proof is given in Appendix Appendix B. Equations (30), (31) demonstrate that differentiation maps elements from set $\mathbb{S}_e$ to higher-order elements of set $\mathbb{S}_o$ and vice versa. This mapping is illustrated in Fig. 5.

The derivative of perfect even solitons has the form (30). Combining (30) with (31) yields that:

$$(x_{2m})''_{tt} = M_2 y_{2(m+2)} \in \mathbb{S}_e, \tag{32}$$

where $x_{2m}, y_{2(m+2)} \in \mathbb{S}_e$. Furthermore, the form (32) is unique for each perfect even soliton x_{2m}.

4. Double-hump soliton solution to a system of non-autonomous Riccati differential equations

In this section, an example of a system of differential equations that admits double-hump soliton solutions is derived using the inverse balancing technique. Let us consider a system of non-autonomous Riccati differential equations coupled with diffusive and multiplicative terms:

$$x'_t = -K(t)\left(a_1 x + a_2 x^2 + a_3 xy + a_4 y\right); \tag{33}$$

$$y'_t = -K(t)\left(b_1 y + b_2 y^2 + b_3 xy + b_4 x\right). \tag{34}$$

The inverse balancing technique is used to determine if ((33), (34)) admits the following solution:

$$x = \sigma_1 M_a(t) + \gamma_1 M_b(t); \quad y = \sigma_2 M_a(t) + \gamma_2 M_b(t); \quad a \neq b. \tag{35}$$

Solution x is a double-hump soliton solution if and only if at least one of the following inequalities holds true:

$$\frac{-\left(\sigma_1+\gamma_1\right)\left(a+2\right)\left(b+2\right)+\sqrt{-\sigma_1\gamma_1\left(a+2\right)\left(b+2\right)\left(a-b\right)^2}}{\gamma_1\left(a+2\right)+\sigma_1\left(b+2\right)}>0; \tag{36}$$

$$\frac{-\left(\sigma_1+\gamma_1\right)\left(a+2\right)\left(b+2\right)-\sqrt{-\sigma_1\gamma_1\left(a+2\right)\left(b+2\right)\left(a-b\right)^2}}{\gamma_1\left(a+2\right)+\sigma_1\left(b+2\right)}>0. \tag{37}$$

Analogous conditions hold for y. Note that it is necessary that the signs of $\sigma_k, \gamma_k, k = 1, 2$ are different for a double-hump solution to exist. Conditions (36), (37) are derived by considering the roots of the derivative of perfect soliton solutions (35).

Let x, y satisfy conditions (36), (37). Then it is clear that (35) is a double-hump soliton solution which has potential applications in the field of optical pulse propagation through optical fibers [12]. The main idea of this technique is to insert (35) into ((33), (34)) and solve the resulting algebraic equations for the parameters of the system. Such derivations have been successfully applied to a number of differential equations [13, 17, 18].

4.1. *Inverse balancing of (35) in ((33), (34))*

Inserting (35) into (33) and using (27) yields:

$$\begin{aligned}
&-K(t)\Big(M_a(t)\left(1+\lambda_a M_a(t)\right)+M_b(t)\left(1+\lambda_b M_b(t)\right)\Big)\\
&=-K(t)\Big(\Big(a_2\sigma_1^2+a_3\sigma_1\sigma_2\Big)M_a^2(t)\\
&+\left(a3_3\sigma_1\gamma_2+a_2\sigma_1\gamma_1+a_3\gamma_1\sigma_2\right)M_a(t)M_b(t)+\Big(a_2\gamma_1^2+a_3\gamma_1\gamma_2\Big)M_b^2(t)\\
&+\left(a_1\sigma_1+a_4\sigma_2\right)M_a(t)+\left(a_1\gamma_1+a_4\gamma_2\right)M_b(t)\Big).
\end{aligned} \tag{38}$$

Cancelling $-K(t)$, moving all terms to the left hand side of (38) and applying (23) transforms (38) into:

$$\begin{aligned}
&-\sigma_1\left(\lambda_a-a_2\sigma_1-a_3\sigma_2\right)M_a^2(t)-\gamma_1\left(\lambda_b-a_2\gamma_1-a_3\gamma_2\right)M_b^2(t)\\
&+\tfrac{1}{a-b}\Big(\left(a_1\sigma_1+a_4\sigma_2-\sigma_1\right)\left(a-b\right)-\left(2+b\right)\left(a_3\sigma_1\gamma_2+\gamma_1\left(2a_2\sigma_1+a_3\sigma_2\right)\right)\Big)M_a(t)\\
&+\tfrac{1}{a-b}\Big(\left(a_1\gamma_1+a_4\gamma_2-\gamma_1\right)\left(a-b\right)+\left(2+a\right)\left(a_3\gamma_1\sigma_2+\sigma_1\left(2a_2\gamma_1+a_3\gamma_2\right)\right)\Big)M_b(t)=0.
\end{aligned} \tag{39}$$

Equating the coefficients of M_a^2, M_b^2, M_a, M_b to zero and solving the resulting linear equations for $a_j, j = 1, \dots, 4$ yields:

$$\begin{aligned} a_1 = \frac{1}{(a-b)\,\Theta}\Bigg(&\Big(\big(2\gamma_1\sigma_2 - \sigma_1\left(\gamma_2-\sigma_2\right)\big)\,b + 4\gamma_1\sigma_2 - 2\sigma_1\left(\gamma_2+\sigma_2\right)\Big)\,a^2 \\ &+ \Big(\big(\left(\gamma_2-\sigma_2\right)\gamma_1 + 2\gamma_2\sigma_1\big)\,b^2 + \left(4\gamma_1\gamma_2 + 4\sigma_1\sigma_2\right)b + 4\left(\gamma_2+\sigma_2\right)\left(\gamma_1-2\sigma_1\right)\Big)\,a \\ &+ \big(4\sigma_1\gamma_2 - 2\gamma_1\left(\gamma_2+\sigma_2\right)\big)\,b^2 - \left(8\gamma_1-4\sigma_1\right)\left(\gamma_2+\sigma_2\right)b - 8\left(\gamma_2+\sigma_2\right)\left(\gamma_1+\sigma_1\right)\Bigg); \end{aligned} \tag{40}$$

$$a_2 = \frac{1}{\Theta}\big(\left(a-2\right)\left(2+b\right)\gamma_2 - \left(b-2\right)\left(2+a\right)\sigma_2\big); \tag{41}$$

$$a_3 = \frac{1}{\Theta}\big(\left(2-a\right)\left(2+b\right)\gamma_1 + \left(b-2\right)\left(2+a\right)\sigma_1\big); \tag{42}$$

$$a_4 = \frac{1}{(b-a)\,\Theta}\big(\left(a-2\right)\left(2+b\right)\gamma_1 + \left(b-2\right)\left(2+a\right)\sigma_1\big)\big(\left(2+b\right)\gamma_1 + \left(2+a\right)\sigma_1\big), \tag{43}$$

where

$$\Theta = \left(2+a\right)\left(2+b\right)\left(\gamma_1\sigma_2 - \gamma_2\sigma_1\right) \neq 0. \tag{44}$$

Analogous computations with respect to (34) yield:

$$\begin{aligned} b_1 = \frac{1}{(a-b)\,\Theta}\Bigg(&\Big(2\gamma_1\sigma_2 + \big(\gamma_1\sigma_2 - \sigma_1\left(2\gamma_2+\sigma_2\right)\big)\,b - \sigma_1\left(4\gamma_2 - 2\sigma_2\right)\Big)\,a^2 \\ &+ \Big(\big(\gamma_2\sigma_1 - \gamma_1\left(\gamma_2+2\sigma_2\right)\big)\,b^2 - \left(4\gamma_1\gamma_2 + 4\sigma_1\sigma_2\right)b - 4\left(\gamma_2-2\sigma_2\right)\left(\gamma_1+\sigma_1\right)\Big)\,a \\ &+ \big(2\gamma_1\left(\gamma_2-2\sigma_2\right) + 2\gamma_2\sigma_1\big)\,b^2 + \left(\gamma_1+\sigma_1\right)\left(8\gamma_2-4\sigma_2\right)b + 8\left(\gamma_2+\sigma_2\right)\left(\gamma_1+\sigma_1\right)\Bigg) \end{aligned} \tag{45}$$

$$b_2 = \frac{1}{\Theta}\big(\left(2-a\right)\left(2+b\right)\gamma_1 + \left(b-2\right)\left(2+a\right)\sigma_1\big); \tag{46}$$

$$b_3 = \frac{1}{\Theta}\big(\left(a-2\right)\left(2+b\right)\gamma_2 - \left(b-2\right)\left(2+a\right)\sigma_2\big); \tag{47}$$

$$b_4 = \frac{1}{(a-b)\,\Theta}\big(\left(2+a\right)\left(2+b\right)\gamma_2 + \left(2+a\right)\left(b-2\right)\sigma_2\big)\big(\left(2+b\right)\gamma_2 + \left(2+a\right)\sigma_2\big). \tag{48}$$

The derivations presented above result in the following Remark.

Remark 3. The necessary existence conditions for perfect soliton solutions (35) in the system ((33),(34)) read:

$$a_2 = b_3; \qquad b_2 = a_3; \tag{49}$$

$$\frac{\sigma_1}{\sigma_2} \neq \frac{\gamma_1}{\gamma_2}. \tag{50}$$

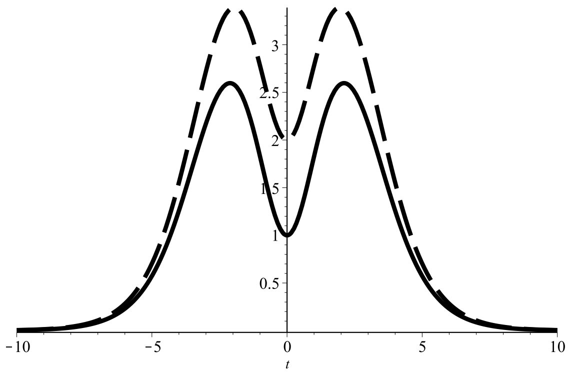

Fig. 6. Double-hump solitons (35) with parameters $a = 20; b = 1; \sigma_1 = 5; \gamma_1 = \gamma_2 = -4; \sigma_2 = 6$. The solid and dashed lines correspond to x and y respectively.

Proof. Condition (49) results from (41), (46) and (42), (47). Observing that $\Theta \neq 0$ in (44) yields condition (50). □

Condition (49) has also been observed in autonomous Riccati systems that admit solitary solutions [17].

4.2. *Computational experiments*

Let us consider double-hump solitons (35) with the following parameters:

$$a = 20;\ b = 1;\ \sigma_1 = 5;\ \gamma_1 = \gamma_2 = -4;\ \sigma_2 = 6. \tag{51}$$

The plot of solitons (35) with parameters (51) is given in Fig. 6.

Inserting parameters (51) into (40)–(48) results in the following system of non-autonomous Riccati differential equations:

$$x'_t = -K(t)\left(\frac{1839}{209}x + \frac{7}{22}x^2 - \frac{53}{132}xy - \frac{7987}{1254}y\right); \tag{52}$$

$$y'_t = -K(t)\left(-\frac{1212}{209}y - \frac{53}{132}y^2 + \frac{7}{22}xy + \frac{1740}{209}x\right). \tag{53}$$

Thus, the obtained systems admits double-hump soliton solutions (35) with parameters (51).

5. Concluding remarks

An analytical framework for the analysis of perfect soliton solutions is presented in this paper. It is demonstrated that solitons expressed as rational functions of exponentials generate a rich variety of solutions, including multi-hump solitons. The properties of classical soliton solutions with imposed symmetry and nonsingularity constraints are investigated. Due to the structure and properties of perfect solitons, the given analytical framework provides a solid foundation for the investigation of multi-hump soliton solutions to nonlinear models describing optical pulse propagations in optical fibers.

A system of Riccati differential equations with multiplicative and diffusive coupling is used to demonstrate the efficiency of the proposed scheme. Note that the inverse balancing technique that is used to illustrate our results is not used to construct the solution to the Riccati system, but to determine necessary existence conditions for multi-hump soliton solutions. A direct approach to the construction of such soliton solutions based on the generalized differential operator technique [16] remains a definite object of future research.

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Appendix A. Derivation of (28)

Using (26) and (27) results in:

$$
\begin{aligned}
(M_a)''_{tt} &= -\left(K'_t M_a\left(1+\lambda_a M_a\right)+K\left(M_a\right)'_t\left(1+\lambda_a M_a\right)+\lambda_a K M_a\left(M_a\right)'_t\right)\\
&= -\frac{1}{2}M_2 M_a\left(1+\lambda_a M_a\right)+K^2 M_a\left(1+\lambda_a M_a\right)^2+\lambda_a K^2 M_a^2\left(1+\lambda_a M_a\right)\\
&= -\frac{1}{2}M_2 M_a\left(1+\lambda_a M_a\right)+K^2 M_a\left(1+\lambda_a M_a\right)\left(1+2\lambda_a M_a\right)\\
&= -\frac{1}{2}M_2 M_a\left(1+\lambda_a M_a\right)+\left(1-M_2\right)M_a\left(1+\lambda_a M_a\right)\left(1+2\lambda_a M_a\right)\\
&= -\frac{1}{2}M_2 M_a\left(1+\lambda_a M_a\right)+\frac{2+a}{4}M_2\left(1-M_a\right)\left(1+\lambda_a M_a\right)\left(1+2\lambda_a M_a\right)\\
&= M_2\left(1+\lambda_a M_a\right)\left(-\frac{1}{2}M_a+\frac{2+a}{4}\left(1-M_a\right)\left(1+2\lambda_a M_a\right)\right)\\
&= \frac{2+a}{4}M_2\left(1+\lambda_a M_a\right)\left(1+\alpha_a^{(1)}M_a\right)\left(1+\alpha_a^{(2)}M_a\right)\\
&= M_a\left(1+\alpha_a^{(1)}M_a\right)\left(1+\alpha_a^{(2)}M_a\right),
\end{aligned}
$$

where

$$
\alpha_a^{(1,2)} = \frac{8-4a}{3a\pm\sqrt{a^2+32}}. \tag{A.1}
$$

The identities

$$M_2\left(1+\frac{2-a}{2+a}M_a\right)=\frac{4}{2+a}M_a; \tag{A.2}$$

$$M_a\left(1+\gamma M_b\right)=\frac{2+a}{2+b}M_b\left(1+\frac{(2+b)\gamma+b-a}{2+a}M_a\right). \tag{A.3}$$

Appendix B. Proof of Equations (30), (31)

Denote $\beta_0 := 0$. Equations (7) and (27) yield:

$$\begin{aligned}(x_{2m})'_t &= \sigma\left(\prod_{j=0}^{m}\left(\beta_j+M_{a_j}\right)\right)'_t = \sigma\sum_{k=0}^{m}\left(\left(M_{a_k}\right)'_t\prod_{\substack{j=0\\ j\neq k}}^{m}\left(\beta_j+M_{a_j}\right)\right)\\ &= \sigma\sum_{k=0}^{m}\left(-KM_{a_k}\left(1+\lambda_{a_k}M_{a_k}\right)\prod_{\substack{j=0\\ j\neq k}}^{m}\left(\beta_j+M_{a_j}\right)\right)\\ &= K\sum_{k=0}^{m}y_{2(m+1),k}=Ky_{2(m+1)},\end{aligned} \tag{B.1}$$

where

$$y_{2(m+1),k}=-\sigma M_{a_k}\left(1+\lambda_{a_k}M_{a_k}\right)\prod_{\substack{j=0\\ j\neq k}}^{m}\left(\beta_j+M_{a_j}\right)\in\mathbb{S}_e, \tag{B.2}$$

and

$$y_{2(m+1)}=\sum_{k=0}^{m}y_{2(m+1),k}. \tag{B.3}$$

Thus (30) holds true. The proof of (31) is analogous.