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Construction of special soliton solutions to the stochastized Riccati equation

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Abstract: A scheme for the analytical stochastization of ordinary differential equations (ODEs) is presented in this paper. Using Itô calculus, an ODE is transformed into a stochastic differential equation (SDE) in such a way that the analytical solutions of the obtained equation can be constructed. Furthermore, the constructed stochastic trajectories remain bounded in the same interval as the deterministic solutions. The presented techniques are applied to the kink soliton solutions of the Riccati differential equation. Obtained stochastic trajectories are generalizations of the well-known kink soliton solutions.

Keywords: stochastic differential equation; analytical solution; Itô calculus

1 Introduction and motivation

Ever since the seminal work by Einstein [1], including and studying the effects of noise on dynamical systems has been an important area of research in physics and other applied sciences. With the introduction of the Wiener process, the derivative of which is white noise, [2] and Itô calculus [3], formal mathematical formulation of stochastic differential equations (SDEs) became possible.

Since then, many areas of applications for SDEs have been the focus of research. One of the first and foremost areas historically areas is economics and finance. SDEs in this area range from the now-classical Black-Scholes equation for call and put option pricing [4] to more recent works. The price change of assets obeying more sophisticated factors than simple supply and demand is modeled via SDEs in [5]. A stochastic differential game of two insurers investing into the same financial market is posed using SDEs in [6]. A duopolistic competition model with sticky prices is considered in [7]. A stochastic control model of investment based on SDEs in presented in [8]. And agent-based model of a financial market is presented in [9].

In recent years, stochastic differential equation models have been applied to a wide array of biological phenomena [10]. A stochastic differential equation model for the evolution of the MCF-7 breast cancer cell line treated by radiotherapy is developed in [11]. Tumor-immune responses to chemotherapy and studied using SDEs in [12]. Plant growth is modeled via a Gompertz-type stochastic differential equation in [13]. Biochemical reaction systems are studied using SDEs in [14].

With the recent outbreak of the novel coronavirus COVID-19, the study of stochastic models for the modeling of this phenomena has emerged. Most are based on the well-known deterministic differential equation model susceptible-exposed-infected-recovered (SEIR) [15, 16]. Adak et al use Brownian motion to induce stochasticity into a SLIR (susceptible-latent-infected-recovered) model by adding stochastic differential directly to a system of ordinary differential equations in [17]. An adaptation of a SIR model to include stochastic transition is discussed in [18]. A SIRS epidemic model including fractional white noise is presented in [19]. A variant of stochastization is used on the generalized logistic equation to model COVID-19 evolution in [20].

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It is clear that the analysis of SDEs is currently a particularly relevant topic. Furthermore, there arises a requirement to induce stochastization into previously deterministic models described via ordinary differential equations (ODEs). While there are many ways to approach this problem, the aim of this paper to provide a technique for stochastization in such a way that if an analytical solution to the ODE can be constructed, an analytical solution to the SDE can also be constructed. Consider the following ODE of the form:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = P(t, y), \tag{1}$$

where $P(t, y)$ is a continuously differentiable function. Let $\omega(t)$ denote a Wiener process [21]. The objective is to construct a stochastic differential equation (SDE) with respect to function $\widetilde{\xi}\left(t, \omega(t)|\alpha\right)$ (where α is a scalar parameter) of the form:

$$\mathrm{d}\,\widetilde{\xi} = h_1\left(t, \widetilde{\xi}\right)\mathrm{d}\,t + h_2\left(t, \widetilde{\xi}\right)\mathrm{d}\,\omega. \tag{2}$$

Equation (2) and its solution possess the following properties:

$$\lim_{\alpha \to 0} \widetilde{\xi}\left(t, \omega(t)|\alpha\right) = y(t); \tag{3}$$

and as α tends to 0, the stochastic differential equation (2) tends to (1) and conversely the solution to (2) tends to the solution of (1).

Note that other approaches to this problem also exist. The most straightforward approach would consist of adding noise to (1) in every step of integration of a numerical integrator. While this technique does enable the randomization of the solution trajectory, a solution of (1) bounded to an interval $I \subset \mathbb{R}$ becomes no longer bounded, which presents a problem for many applications.

Another option for inputting randomness into an ODE is based on random differential equations, discussed in detail in [22]. Here the time variable t is replaced by a stochastic Wiener process $\omega(t)$, yielding the following random differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = P\left(\omega(t), y\right). \tag{4}$$

The latter approach is an improvement boundedness-wise over the former, however, the menagerie of trajectories obtained from (4) would be much smaller than those from (2).

The main objective of this paper is to construct a semi-analytical scheme for solving (2), and to apply this scheme to the paradigmatic Riccati equation [23]. It is well known that the Riccati equation does admit the first order solitary solution (the kink-type solution) [24]. In its turn, soliton solutions (and the Riccati equation in general) do play an important role in defining the global dynamics of different models, including the prostate cancer model [25], the Hepatitis C treatment model [26], COVID-19 within host model with immune response [27].

Stochastization of such models describing different biological and biomedical processes is an important research direction [17, 28]. Stochastic models allow a better description of real-life processes and do help to represent the effect of local unpredictability caused by the noise and different uncertainties. However, computational analysis of stochastic differential equations containing a variable which represents random white noise (usually calculated as the derivative of Brownian motion or the Wiener process) results into the investigation of martingales [29]. The matter of fact is that a particular solution of a stochastic differential equation can wander far away from its non-stochastic counterpart solution as the time moves away from the initial conditions [30].

However, one needs to keep in mind that the range of values of the solution (especially if it is a soliton solution) is predetermined by the structure of the model (for example, the concentration of the infected cells cannot be negative). Therefore, it is important to construct such stochastization schemes for ordinary differential equations which would guarantee that the stochastic soliton solution would remain in the predetermined range of values. The derivation of such soliton solutions to the stochastic Riccati equation is the main objective of this paper.

This paper is organized as follows. Preliminary results of Itô calculus and some operator methods for the solution of differential equations are presented in Section 2; the inverse balancing technique is adapted to stochastic differential equations in Section 3; the scheme for the stochastization of first-order ODEs in constructed in Section 4; the developed scheme is applied to obtain the stochastization of the Riccati equation in Section 5; concluding remarks are given in Section 6.

2 Preliminaries

A short review of the concepts of Itô calculus required for the paper's main results is provided in this section.

2.1 Wiener process and Itô integral

Let us consider a non-differentiable Wiener process $\omega(t)$ with the following properties [21]:

$$\lim_{\Delta t \to 0} \Delta\omega(t) = 0; \tag{5}$$

$$\lim_{\Delta t \to 0} \frac{\big(\Delta\omega(t)\big)^n}{\Delta t} = \begin{cases} 1, & n = 2 \\ 0, & n = 3, 4, \dots \end{cases} \tag{6}$$

Let $\sigma(t, x)$ be continuous and satisfy a global Lipschitz condition. The integral of $\sigma\left(t, \omega(t)\right)$ with respect to the Wiener process is defined as the Itô integral [22]:

$$\begin{aligned} I\left(t, \omega(t)\right) &= \int_0^t a\left(s, \omega(s)\right) \mathrm{d}\, t + \int_0^t \sigma\left(s, \omega(s)\right) \mathrm{d}\, \omega(s) \\ &= \lim_{N \to +\infty} \sum_{j=0}^{N-1} a\left(t_j^{(N)}, \omega\left(t_j^{(N)}\right)\right)\left(t_{j+1}^{(N)} - t_j^{(N)}\right) \\ &+ \lim_{N \to +\infty} \sum_{j=0}^{N-1} \sigma\left(t_j^{(N)}, \omega\left(t_j^{(N)}\right)\right)\left(\omega\left(t_{j+1}^{(N)}\right) - \omega\left(t_j^{(N)}\right)\right), \end{aligned} \tag{7}$$

where $0 = t_0^{(N)} < t_1^{(N)} < \dots < t_N^{(N)} = t$ is a partition of the interval $[0, t]$. The above limit in taken in the mean-square sense, ensuring that $\max\limits_{0 \le j \le N-1} \left(t_{j+1}^{(N)} - t_j^{(N)}\right) \to 0$.

2.2 Itô's lemma

Itô's lemma is a fundamental result in SDE theory. Suppose a process $\xi(t)$ is given that has the following differential:

$$\mathrm{d}\, \xi(t) = a_\xi(t, \xi)\mathrm{d}\, t + \sigma_\xi(t, \xi)\mathrm{d}\, \omega(t), \tag{8}$$

where a_ξ, σ_ξ and an additional function $f(t, x)$ satisfy conditions detailed in the previous section. Then, the differential of process $f\left(t, \xi\right)$ is given by:

$$\mathrm{d}\, f(t, \xi) = \left(\frac{\partial f}{\partial t} + a_\xi \frac{\partial f}{\partial x} + \frac{1}{2}\sigma_\xi^2 \frac{\partial^2 f}{\partial x^2}\right)\Bigg|_{x=\xi} \mathrm{d}\, t + \sigma_\xi \frac{\partial f}{\partial x}\Bigg|_{x=\xi} \mathrm{d}\, \omega(t). \tag{9}$$

The above equation is an analogy of the chain rule of differentiation for stochastic processes.

2.3 Stochastic differential equations and their solutions

Firstly, consider the following stochastic equation where the coefficients depend only on t:

$$\mathrm{d}\, \eta = a_\eta(t)\mathrm{d}\, t + \sigma_\eta(t)\mathrm{d}\, \omega(t). \tag{10}$$

2.5 Operator solution to a particular partial differential equation

In subsequent sections, a partial differential equation arises that possesses the following form:

$$\frac{\partial u}{\partial t} - f_0(t,x)\frac{\partial u}{\partial x} = f_1(t,x)u(t,x), \tag{22}$$

where $f_0(t,x), f_1(t,x)$ are analytical functions. The following boundary condition is posed:

$$u\,(0,x) = \varphi_0(x), \tag{23}$$

where $\varphi_0(x)$ is also analytic.

An operator solution to (22), (23) is presented in [32]. Let $\mathbf{L}_\lambda$ denote the integration operator with respect to variable λ.

Then, the solution to (22), (23) is given by:

$$u(t,x) = \exp\big(A(t,x)\big)\,, \tag{24}$$

where $A(t,x)$ is the solution to the following operator problem:

$$\big(\mathbf{D}_t - f_0(t,x)\mathbf{D}_x\big)\,A = f_1(t,x); \tag{25}$$

$$A(0,x) = \ln\varphi_0(x) = \psi_0(x). \tag{26}$$

The solution to (25), (26) reads:

$$A(t,x) = \left(\sum_{k=0}^{+\infty}\big(\mathbf{L}_t f_0(t,x)\mathbf{D}_x\big)^k\right)\big(\mathbf{L}_t f_1(t,x) + \psi_0(x)\big)\,. \tag{27}$$

3 Inverse balancing technique for SDEs

The main idea of the inverse balancing technique is as follows: given a differential equation, and an analytical form of its solution, to assume that the solution is known, and determine the parameters of the differential equation in terms of the solution parameters. This yields a robust approach to determining necessary conditions of existence of a particular solution to a class of differential equations. It has been applied in a variety of fields, ranging from astrophysics [35], population dynamics [36] to medicine [25].

In this section, the method is extended to include stochastic differential equations.

Theorem 3.1. *Suppose that the Itô function $\Phi_\xi(t)$, $\sigma_\xi\,(t,x)$ and a constant $0 < \alpha \leq 1$ are given.*

Then, the Itô partial differential equation with respect to function $a_\xi\,\big(t,x|\alpha\big)$

$$\sigma_\xi(t,x)\left(\frac{1}{\sigma_\xi^2\,(t,x)}\frac{\partial\sigma_\xi(t,x)}{\partial t} - \frac{\partial}{\partial x}\frac{a_\xi\,\big(t,x|\alpha\big)}{\sigma_\xi\,(t,x)} + \frac{\alpha^2}{2}\frac{\partial^2\sigma_\xi(t,x)}{\partial x^2}\right) = \Phi_\xi(t); \tag{28}$$

has the following general solution:

$$\begin{aligned} a_\xi\,\big(t,x|\alpha\big) = \sigma_\xi(t,x)\Bigg(&-\int_0^x \frac{\partial}{\partial t}\left(\frac{1}{\sigma_\xi\,(t,u)}\right)\mathrm{d}\,u - \Phi_\xi(t)\int_0^x \frac{\mathrm{d}\,u}{\sigma_\xi(t,u)} \\ &+ \frac{\alpha^2}{2}\frac{\partial\sigma_\xi(t,x)}{\partial x} + \frac{C(t)}{\sigma_\xi(t,0)} - \frac{\alpha^2}{2}\left.\frac{\partial\sigma_\xi(t,x)}{\partial x}\right|_{x=0}\Bigg), \end{aligned} \tag{29}$$

where $C(t)$ is an arbitrary function.

Theorem 3.2. *The stochastic differential equation with respect to process $\widetilde{\xi} = \widetilde{\xi}\left(t, \omega(t)|\alpha\right)$:*

$$\mathrm{d}\,\widetilde{\xi} = \left(P\left(t,\widetilde{\xi}\right) + \alpha Q\left(t,\widetilde{\xi}|\alpha\right)\right)\mathrm{d}\,t + \alpha\sigma_{\xi}\left(t,\widetilde{\xi}\right)\mathrm{d}\,\omega(t); \tag{40}$$

satisfies the Itô condition (13) with Itô function $\Phi_{\widetilde{\xi}}(t) = \Phi_{\xi}(t)$, which is defined by (28).

Proof. Note that

$$a_{\widetilde{\xi}}\left(t,\widetilde{\xi}|\alpha\right) = P\left(t,\widetilde{\xi}\right) + \alpha Q\left(t,\widetilde{\xi}|\alpha\right); \tag{41}$$

$$\sigma_{\widetilde{\xi}}\left(t,\widetilde{\xi}|\alpha\right) = \alpha\sigma_{\xi}\left(t,\widetilde{\xi}\right). \tag{42}$$

The Itô condition for these functions is given by (28), which results in the proof of the theorem. □

Taking the limit as $\alpha \to 0$ in the SDE (40) results in the ODE (1). Note that the solution of the SDE also tends to the deterministic solution of the ODE:

$$\lim_{\alpha\to 0}\widetilde{\xi}\left(t,\omega(t)\mid\alpha\right) = \widetilde{\xi}\left(t,0\right) = y(t). \tag{43}$$

4 Stochastization of first-order ODEs

4.1 Construction of analytical solutions to (40)

In order to construct the analytical solutions to (40) the algorithm described in section 2.3 is applied. Let (40) be given. A transformation $\widetilde{\eta}(t) = f(t,\widetilde{\xi})$ of process $\widetilde{\xi}$ must be determined in order to transform (40) into:

$$\mathrm{d}\,\widetilde{\eta} = a_{\widetilde{\eta}}(t|\alpha)\mathrm{d}\,t + \sigma_{\widetilde{\eta}}(t)\mathrm{d}\,\omega(t). \tag{44}$$

The solution to (44) is given by (11), which results in $\widetilde{\xi} = g(t,\widetilde{\eta}(t))$, where $g(t,x)$ is the inverse transformation to $f(t,x)$ with respect to x.

Theorem 4.1. *Functions $a_{\widetilde{\eta}}(t), \sigma_{\widetilde{\eta}}(t)$ are given by:*

$$a_{\widetilde{\eta}}(t|\alpha) = \sigma_{\widetilde{\eta}}(t)\left(\frac{1}{\alpha}S_{\xi}(t) + S_{\xi}^{(+)}(t)\right); \tag{45}$$

$$\sigma_{\widetilde{\eta}}(t) = \gamma\exp\left(\int_0^t \Phi_{\widetilde{\xi}}(s)\mathrm{d}\,s\right); \quad \gamma\in\mathbb{R}\setminus\{0\}, \tag{46}$$

where $S_{\xi}(t), S_{\xi}^{(+)}(t)$ are defined by (39).

Proof. Using (16) directly yields (46). Then, Itô lemma (9) yields:

$$a_{\widetilde{\eta}}(t|\alpha) = \frac{\partial f}{\partial t} + a_{\widetilde{\xi}}\left(t,x|\alpha\right)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma_{\widetilde{\xi}}\left(t,x|\alpha\right)^2\frac{\partial^2 f}{\partial x^2}. \tag{47}$$

Inserting (15) into (47) results in:

$$a_{\widetilde{\eta}}(t|\alpha) = \frac{\partial}{\partial t}\left(\sigma_{\widetilde{\eta}}(t)\int_0^x \frac{\mathrm{d}\,u}{\sigma_{\widetilde{\xi}}(t,u|\alpha)}\right) + a_{\widetilde{\xi}}(t,x|\alpha)\frac{\sigma_{\widetilde{\eta}}(t)}{\sigma_{\widetilde{\xi}}(t,x|\alpha)} - \frac{\sigma_{\widetilde{\eta}}(t)}{2}\frac{\partial\sigma_{\widetilde{\xi}}(t,x|\alpha)}{\partial x}. \tag{48}$$

Let $\Psi(t, z)$ satisfy the following:

$$\int_0^{\Psi} \frac{\mathrm{d}\,u}{\sigma_\xi(t,u)} = z. \tag{58}$$

Differentiating (58) with respect to Ψ and rearranging yields:

$$\frac{\partial \Psi}{\partial z} = \sigma_\xi\left(t, \Psi\right); \quad \Psi(t,0) = 0. \tag{59}$$

Applying (59) to (57) yields (53). □

Corollary 4.1. *The solution to SDE (40) is given by:*

$$\widetilde{\xi}\left(t, \omega(t)|\alpha\right) = g\left(t, \widetilde{\eta}\left(t|\alpha\right)\right), \tag{60}$$

where $g, \widetilde{\eta}$ are given by (53) and (51) respectively.

4.2 Scheme for stochastization of first-order ODEs

Suppose that the ODE (1) and an Itô function $\Phi_{\widetilde{\xi}}(t)$ are given. Thus, the function $P(t,x)$ is known and (37) yields:

$$S_\xi(t) = \frac{P(t,x)}{\sigma_\xi(t,x)} + \int_0^x \left(\frac{\Phi_{\widetilde{\xi}}(t)}{\sigma_\xi(t,u)} + \frac{\partial}{\partial t}\frac{1}{\sigma_\xi(t,u)}\right) \mathrm{d}\,u. \tag{61}$$

Note that the left hand side of (61) does not depend on x, thus differentiating (61) with respect to x results in a partial differential equation with respect to the unknown function $\sigma_\xi(t,x)$:

$$\frac{\partial \sigma_\xi}{\partial t} + P\frac{\partial \sigma_\xi}{\partial x} = \sigma_\xi(t,x)\left(\Phi_{\widetilde{\xi}}(t) + \frac{\partial P}{\partial x}\right). \tag{62}$$

The above differential equation can be used to determine such $\sigma_\xi(t,x)$ that (61) holds true. Then, $Q(t,x)$ can be computed via (38), leading to (40) which is the stochastization of (1).

5 Stochastization of the Riccati equation

5.1 General case

Consider the Riccati differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \delta\left(y - y_1\right)\left(y - y_2\right); \quad \delta, y_1, y_2 \in \mathbb{R}. \tag{63}$$

The aim of this section is to provide a stochastization of (63) in the form (40).

Let $P(x) = \delta\left(x - y_1\right)\left(x - y_2\right)$. Note that (62) has the form (22) with functions $f_0(t,x) = -P(x)$, $f_1(t,x) = \Phi_{\widetilde{\xi}}(t) + \frac{\mathrm{d}P}{\mathrm{d}x}$. Then, following from section 2.5, the solution to (62) reads:

$$\sigma_\xi(t,x) = \exp\left(A(t,x)\right), \tag{64}$$

where, by (27):

$$A(t,x) = \left(\sum_{k=0}^{+\infty}\left(-\mathbf{L}_t P(x)\mathbf{D}_x\right)^k\right)\left(\mathbf{L}_t\left(\Phi_{\widetilde{\xi}}(t) + \frac{\mathrm{d}P}{\mathrm{d}x}\right) + \psi_0(x)\right), \tag{65}$$

and the function $\psi_0(x)$ satisfies $\sigma_\xi(0,x) = \exp\psi_0(x)$. Note that when considering the stochastization of an ODE, $\psi_0(x)$ can be selected arbitrarily.

where $y(t,x)$ is given by (69). The solution the functional equation (74) with respect to $B(x)$ reads:

$$B(x) = \ln\left(\nu\delta\left(x-y_1\right)\left(x-y_2\right)\right) = \ln\left(\nu P(x)\right); \quad \nu \in \mathbb{R}\backslash\{0\}, \tag{75}$$

when $\widehat{\Phi}_{\widetilde{\xi}}(t) = \Phi_{\widetilde{\xi}}(t) \equiv 0$. In that case, $\sigma_{\widetilde{\xi}}(t,x)$ has the following form:

$$\sigma_{\widetilde{\xi}}(t,x) = \nu\delta\left(x-y_1\right)\left(x-y_2\right) = \nu P(x). \tag{76}$$

Inserting (76) into (71) yields the stochastization of (63) in the special case:

$$\mathrm{d}\,\widetilde{\xi} = P\left(\widetilde{\xi}\right)\left(1+\frac{\nu\alpha^2}{2}P\left(\widetilde{\xi}\right)\left(\nu\delta\left(2\widetilde{\xi}-y_1-y_2\right)+S_{\widetilde{\xi}}^{(+)}\right)\mathrm{d}\,t+\alpha\nu\mathrm{d}\,\omega(t)\right), \tag{77}$$

where $S_{\widetilde{\xi}}^{(+)} = \ln\left(\nu\delta y_1 y_2\right)\frac{y_1+y_2}{y_1 y_2}$.

5.3 Analytical solution of (77)

Following the algorithm outlined in Theorem 4.1, the functions (45) and (46) read:

$$a_{\widetilde{\eta}}(t|\alpha) = \gamma\left(\frac{\nu}{\alpha}+S_{\widetilde{\xi}}^{(+)}\right); \tag{78}$$

$$\sigma_{\widetilde{\eta}}(t) = \gamma; \quad \gamma \in \mathbb{R}\backslash\{0\}. \tag{79}$$

Thus, (77) is transformed into:

$$\mathrm{d}\,\widetilde{\eta} = \gamma\left(\frac{\nu}{\alpha}+S_{\widetilde{\xi}}^{(+)}\right)\mathrm{d}\,t+\gamma\mathrm{d}\,\omega(t), \tag{80}$$

with the solution

$$\widetilde{\eta}(t|\alpha) = \widetilde{\eta}_0+\gamma\left(\frac{\nu}{\alpha}+S_{\widetilde{\xi}}^{(+)}\right)t+\gamma\omega(t). \tag{81}$$

By Theorem 4.2, the function $\Psi(z)$ must be derived from the differential equation:

$$\frac{\mathrm{d}\Psi}{\mathrm{d}z} = \nu\delta(z-y_1)(z-y_2); \quad \Psi(0)=0. \tag{82}$$

The solution to the above equation reads:

$$\Psi(z) = y_2\frac{\exp\left(\kappa z\right)-1}{\exp\left(\kappa z\right)-\frac{y_2}{y_1}}, \quad \kappa = \delta\nu(y_1-y_2). \tag{83}$$

By (53) we obtain

$$g(t,x) = g(x) = \Psi\left(\frac{\alpha}{\gamma}x\right), \tag{84}$$

and the analytical solution to (77) is given by:

$$\begin{aligned}\widetilde{\xi}\left(t|\alpha\right) &= g\left(\widetilde{\eta}(t|\alpha)\right)\\ &= y_2\frac{\exp\left(\kappa\frac{\alpha\nu}{\gamma}\left(\widetilde{\eta}_0+\gamma\left(\frac{\nu}{\alpha}+S_{\widetilde{\xi}}^{(+)}\right)t+\gamma\omega(t)\right)\right)-1}{\exp\left(\kappa\frac{\alpha\nu}{\gamma}\left(\widetilde{\eta}_0+\gamma\left(\frac{\nu}{\alpha}+S_{\widetilde{\xi}}^{(+)}\right)t+\gamma\omega(t)\right)\right)-\frac{y_2}{y_1}}.\end{aligned} \tag{85}$$

Since it must hold that $\lim_{\alpha\to 0}\widetilde{\xi}(t|\alpha) = y(t)$, where $y(t)$ is given by (69), parameters $\widetilde{\eta}_0 = 0, \nu = 1$. Inserting these values into (85) yields:

$$\widetilde{\xi}(t|\alpha) = y_2\frac{\exp\left(\kappa t+\kappa\alpha\left(S_{\widetilde{\xi}}^{(+)}t+\omega(t)\right)\right)-1}{\exp\left(\kappa t+\kappa\alpha\left(S_{\widetilde{\xi}}^{(+)}t+\omega(t)\right)\right)-\frac{y_2}{y_1}}. \tag{86}$$

Note that comparing the above solution to (69) it can be seen that only the variable within the exp function has the Wiener process $\omega(t)$. This means that the solution of the stochastized Riccati equation belongs to the same set of values as the deterministic Riccati equation.

5.4 Numerical comparison: stochastization and randomization of the Riccati equation

In this section, two different approaches to induce randomness into the Riccati equation are compared.

The first approach is described in sections 5.2, 5.3, which leads to a special case of the stochastized Riccati equation (77) and its analytical solution (86).

The randomization procedure is described as follows. Let the Riccati equation (63), a scaling variable $\varepsilon > 0$ and a sample $\theta_0, \ldots, \theta_n$ of a Gaussian random variable with mean zero and unit variance be given. Consider any constant-step time-forward numerical integrator with step size $h > 0$. The randomized solution at points $t_k = kh, k = 0, \ldots, n$ is denoted as $\widehat{\xi}_k = \widehat{\xi}\,(t_k)$.

We initialize the process by setting the first point equal to the initial condition: $\widehat{\xi}_0 = y_0$. In the k-th step, the value $\widehat{\xi}_k$ is computed by performing one time-forward integration step for following the differential equation:

$$\frac{\mathrm{d}\widehat{\xi}}{\mathrm{d}t} = \delta\left(\widehat{\xi} - y_1\right)\left(\widehat{\xi} - y_2\right) + \varepsilon\theta_k; \quad \widehat{\xi}\left(t_{k-1}\right) = \widehat{\xi}_{k-1},\ k = 1, \ldots, n. \tag{87}$$

The process of randomization described above yields a random solution trajectory with the mean close to the deterministic solution (see Fig. 1 (a)). However, note that the randomized solution $\widehat{\xi}$ leaves the interval to which the deterministic solution $y(t)$ is bound. This not the case for the stochastized solution, which remains bounded to the same interval as $y(t)$ (see Fig. 1 (b)).

Note that the statistical mean of function (86) is not equal to the solution (69) of the deterministic Riccati equation. As mentioned before, the deterministic Riccati equation solution is obtained when $\alpha \to 0$.

6 Concluding remarks

A scheme for the analytical stochastization of ODEs is presented in this paper. Given an ODE, its stochastic differential equation counterpart is constructed in such a way that it satisfies the Itô condition. This ensures that it is possible to construct an analytical solution to the obtained SDE via the application of Itô calculus.

The described technique of stochastization has two important properties: as the parameter α that governs the influence of randomness on the SDE solution tends to zero, the solution tends to the deterministic ODE solution. Furthermore, if the ODE solution is bounded to an interval, the constructed stochastic trajectories can also only belong to that interval – which is not true for most other stochastization schemes.

The scheme is applied to the paradigmatic Riccati equation which possesses kink soliton solutions. It is shown that the general analytical form of the deterministic solution is preserved in the stochastic solution after the transformation of the ODE to the SDE. Stochastic trajectories obtained in this manner are a generalization of kink soliton solutions in the stochastic sense.

The extension of the presented stochastization scheme to higher-order ODEs and systems of ODEs, as well as applications to real-world models, remains a definite objective of future research.

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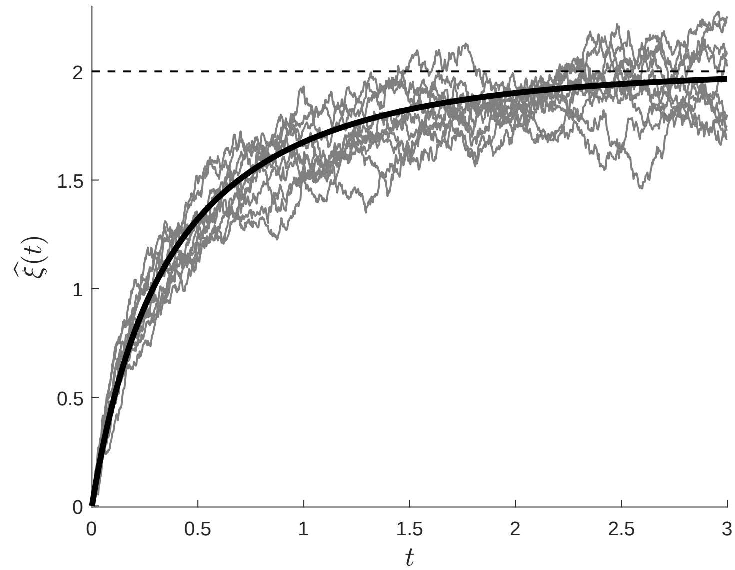
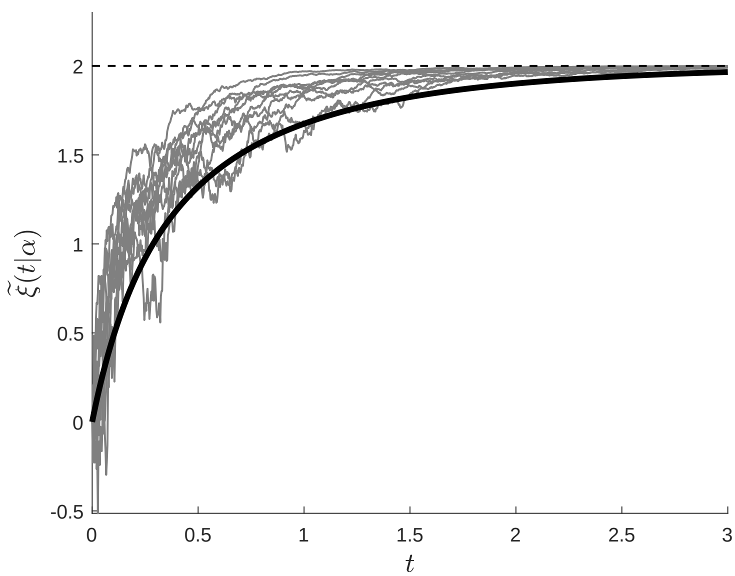


(a)



(b)

Figure 1. The randomized solution $\widehat{\xi}(t)$ of the Riccati equation (part (a)) and the solution $\widetilde{\xi}(t|\alpha)$ of the stochastized Riccati equation (part (b)). Parameters of the Riccati equation are set to $\delta = 1, y_1 = 2, y_2 = 3$; the initial conditions are set to zero at $t = 0$. The scaling variable ε is set to 5 in (a); α is set to 0.5 in (b). Thin grey lines denote randomized and stochastized solution trajectories in (a) and (b) respectively. Thick black lines depict the solution of the deterministic Riccati equation. The dashed black line denotes the upper bound for the deterministic solution for $t > 0$.