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Higher order solitary solutions to the meta-model of diffusively coupled Lotka-Volterra systems
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Higher order solitary solutions to the meta-model of diffusively coupled Lotka-Volterra systems

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Abstract A meta-model of diffusively coupled Lotka-Volterra systems used to model various biomedical phenomena is considered in this paper. Necessary and sufficient conditions for the existence of nth order solitary solutions are derived via a modified inverse balancing technique. It is shown that as the highest possible solitary solution order n is increased, the number of non-zero solution parameter values remains constant for solitary solutions of order $n > 3$. Analytical and computational experiments are used to illustrate the obtained results.

Keywords nonlinear differential equation · analytical solution · COVID model

Mathematics Subject Classification (2010) 35C08 · 34A25 · 34D06

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1 Introduction

Even though solitons (also called solitary solutions) have been first discovered in the 19th century by John Scott Russell [34], later formalized by Korteweg and de Vries [16] and made famous in the mid-20th century by the Fermi-Pasta-Ulam computational experiment [12] and later works by Norman and Zabusky [42], they remain at the forefront of research to this day due to their unique physical properties.

In recent years, the emergence of powerful computer algebra software coupled with a marked rise in computing power has sparked a new interest in the subject. Analytical investigation is central to the construction of solitary solutions and availability of aforementioned tools has greatly increased the number of studies in this area.

Classical methods used to construct solitary solutions to differential equations include the inverse scattering transform [2], the Bäcklund transform [33] and the Darboux transform [32] methods.

More recently developed techniques that make use of computer algebra software include the simplest equation method and its extensions [17,19], the equivalent (G'/G)-extension and tanh-extension methods [41,18] and the homotopy analysis method [1].

The determination of solitary solutions to differential equations is an important question in applied research. Some recent examples are given below. Soliton crystals have been observed and characterized in monolothic Kerr resonators which offers a novel way to increase the efficiency of Kerr combs [8]. A generalized hydrodynamics theory based on soliton solutions is developed in [10] and is applied to the Lieb-Liniger model realized in cold-atom experiments. Solitons have been observed in the cell movement of a cellular slime mould in [13]. Biological population models have been shown to possess solitary solutions on multiple occasions, including [30,3].

The motivation for this study and its contributions to the theory of solitary solutions is given in the next section.

2 Motivation

2.1 The diffusive and the multiplicative coupling

Mathematical modelling of interacting dynamical systems is a classical field of research. For example, a diffusive coupling between two (or more) dynamical systems is used to model the effect of synchronization. The paradigmatic model of two diffusively coupled dynamics is described by:

$$x'_t = F(x) + \gamma(y - x); \tag{1}$$

$$y'_t = F(y) + \gamma(x - y), \tag{2}$$

where F is the vector field modelling the isolated chaotic dynamics; t is time; γ is the diffusive coupling parameter (usually set a positive constant). The

2.2 The multiplicative coupling in SEIR and cancer tumor growth models

The recent COVID-19 coronavirus outbreak caused a huge interest in mathematical models capable to describe the spread of pandemics. The SEIR (Susceptible - Exposed - Infectious Recovered) compartmental model [14] and a large variety of its modifications (SEIRS, SEIRV) play an important role in the mathematical description of pandemic spread in different regions [35, 4, 15]. The paradigmatic SEIR model reads:

$$S'_t = -\frac{\beta}{N} SI; \tag{9}$$

$$E'_t = \frac{\beta}{N} SI - \sigma E; \tag{10}$$

$$I'_t = \sigma E - \gamma I; \tag{11}$$

$$R'_t = \gamma I, \tag{12}$$

where $N = S + E + I + R$ is the total population; β is the infectious rate; σ is the incubation rate; γ is the recovery rate. The dynamics of the SEIR model is well understood and investigated in many different theoretical and practical aspects.

It is worth noting that the SEIR model (9)-(12) reduces to a linear system of ordinary differential equations when the infection rate β tends to zero. Such a reduction (based on the elimination of the multiplicative coupling terms) yields linear systems also for SEIRS, SEIRV, SEIRS with vital dynamics and many other models based on the SEIR nomenclature [44, 9].

The analogy between the Lotka-Volterra model (3)-(4) and the competitive Lotka-Volterra model (6)-(7) suggests the following model with the multiplicative coupling:

$$x'_t = a_{0x} + a_{1x}x + a_{2x}x^2 + \beta_{xy}xy; \tag{13}$$

$$y'_t = b_{0y} + b_{1y}x + b_{2y}x^2 + \beta_{yx}xy, \tag{14}$$

where $a_{2x}, b_{2y} \neq 0$. The system (13)-(14) reduces to two uncoupled Riccati equations with constant coefficients when the coupling coefficients β_{xy} and β_{yx} vanish to zero. It appears that such models are widely used to describe the interaction between healthy and cancer cells in phenomenological mathematical models of a single cancer tumor [20]. Such types of models comprising two Riccati type equations coupled with multiplicative terms are used for the description of prostate cancer treatment with androgen deprivation therapy [43], cancer stem cell- targeted immunotherapy [38], the maximization of viability time in general cancer therapy [7].

2.3 The meta-model of coupled prey-predator systems

The mathematical meta-model of diffusively coupled Lotka-Volterra systems on heterogenous graphs in presented in [22]. When the number of systems is limited to two, the model of diffusively coupled predator-prey systems reads [22]:

$$x_1' = a_{11}x_1 - \lambda_1 x_1 y_1; \quad (15)$$
$$y_1' = b_{11}y_1 - \mu_1 x_1 y_1 + \gamma_1(y_2 - y_1); \quad (16)$$
$$x_2' = a_{12}x_2 - \lambda_2 x_2 y_2; \quad (17)$$
$$y_2' = b_{12}y_2 - \mu_2 x_2 y_2 + \gamma_2(y_1 - y_2). \quad (18)$$
$$(19)$$

The system of diffusively coupled Lotka-Volterra models (15)-(18) reduces into a system of linear coupled differential equations when the multiplicative coupling constants λ_1, μ_1, λ_2, μ_2 vanish to zero (in analogy to (9)-(12)). A natural extension of (15)-(18) is based on the expansion of the basic Lotka-Volterra model by the nonlinear terms (in accordance to (6)-(7)):

$$x_1' = a_{01} + a_{11}x_1 + a_{21}x_1^2 + \lambda_1 x_1 y_1; \quad (20)$$
$$y_1' = b_{01} + b_{11}y_1 + b_{21}y_1^2 + \mu_1 x_1 y_1 + \gamma_1 (y_2 - y_1); \quad (21)$$
$$x_2' = a_{02} + a_{12}x_2 + a_{22}x_2^2 + \lambda_2 x_2 y_2; \quad (22)$$
$$y_2' = b_{02} + b_{12}y_2 + b_{22}y_2^2 + \mu_2 x_2 y_2 + \gamma_2 (y_1 - y_2). \quad (23)$$

System (20)-(23) does represent a meta-model of two diffusively coupled Riccati systems (each system comprises two Riccati equations coupled with multiplicative terms). System (20)-(23) splits into two uncoupled systems described by (13)-(14) when the diffusive coupling constants γ_1 and γ_2 vanish to zero. Analogously, system (20)-(23) splits into four uncoupled Riccati equations (8) when both the diffusive and the multiplicative coupling constants vanish to zero. In other words, the model described by (20)-(23) generalizes the competitive Lotka-Volterra model in the spatial domain.

In fact, such a diffusive coupling between two Riccati systems coupled with multiplicative terms is already used to describe the phenomenological model of metastasis comprising two interacting tumors [31].

2.4 The motivation of this paper

The existence of the first order soliton-type solutions (kink solitons) to Riccati equation (8) is known for decades [29]. Necessary and sufficient conditions for the existence of second-order soliton-type solutions (dark/bright solitons) to system (13)-(14) has been recently reported in [24]. The existence of n-th order soliton-type solutions to the meta-model of coupled Riccati equations (20)-(23) poses a serious challenge from the mathematical point of view. Providing two answers to the following questions what the maximal order n is, and what are the necessary and sufficient conditions for the existence of solitons up to the n-th order is the main objective of this paper.

3 Preliminaries

3.1 Definition of the solitary solution

Solitary solutions of the following form [36,23] are considered in this paper:

$$x(t) = \sigma \frac{\prod_{k=1}^{n} \left(\exp(\eta(t-t_0)) - x_k\right)}{\prod_{k=1}^{n} \left(\exp(\eta(t-t_0)) - t_k\right)}, \tag{24}$$

where $n \in \mathbb{N}$ - order of the solitary solution, $t_0, \sigma, \eta \in \mathbb{R}$, $x_k, t_k \in \mathbb{C}$.

The following independent variable transformation is introduced:

$$\widehat{t} = \exp(\eta(t-t_0)). \tag{25}$$

Using (25) on (24) simplifies the analytical expression of the solitary solution (24) as follows:

$$x(t) = x\left(\frac{\ln \widehat{t}}{\eta} + t_0\right) = \widehat{x}\left(\widehat{t}\right) = \widehat{x} = \sigma \frac{X\left(\widehat{t}\right)}{T\left(\widehat{t}\right)}, \tag{26}$$

where

$$X(\theta) = \prod_{k=1}^{n} (\theta - x_k)\,; \quad T(\theta) = \prod_{k=1}^{n} (\theta - t_k)\,. \tag{27}$$

3.2 Solitary solutions to Riccati equations

3.2.1 Uncoupled Riccati equations

Consider the following Riccati equation with respect to $x = x(t)$:

$$x' = c_0 + c_1 x + c_2 x^2, \tag{28}$$

where $c_0, c_1, c_2 \in \mathbb{C}$.

Equation (28) can be transformed via the substitution (25), where $\eta^2 = c_1^2 - 4c_0c_2$ [29], as follows:

$$\eta \widehat{t} \widehat{x}'_{\widehat{t}} = c_0 + c_1 \widehat{x} + c_2 \widehat{x}^2. \tag{29}$$

The solution to (29) reads [29]:

$$\widehat{x} = \sigma \frac{\widehat{t} - s x_0}{\widehat{t} - s t_0} = \sigma \frac{\widehat{t} - \frac{x_0}{t_0} æ_0}{\widehat{t} - æ_0}, \tag{30}$$

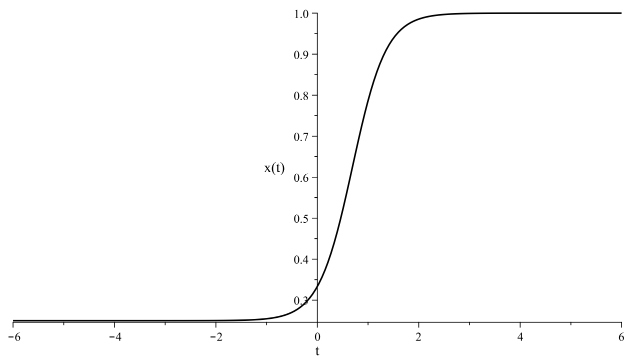


Fig. 1 Kink solitary solution to (35) with initial condition $x(0) = \frac{1}{3}$.

where $s \in \mathbb{R}$ is a free constant, $æ_0 = st_0$ and parameters σ, x_0, t_0 satisfy the following identities:

$$c_0 = \frac{\sigma x_0 \eta}{x_0 - t_0}; \tag{31}$$

$$c_1 = \frac{(t_0 + x_0)\,\eta}{t_0 - x_0}; \tag{32}$$

$$c_2 = \frac{t_0 \eta}{\sigma\,(x_0 - t_0)}. \tag{33}$$

Thus the solution to (28) reads:

$$x = \sigma \frac{\exp(\eta t) - s x_0}{\exp(\eta t) - s t_0}. \tag{34}$$

This solution is known as the kink solitary solution [37]. It describes the transition of a system from one steady state to another via a monotonous trajectory.

Example Suppose that the following Ricatti differential equation with respect to $x = x(t)$ is given:

$$x' = -1 + 5x - 4x^2. \tag{35}$$

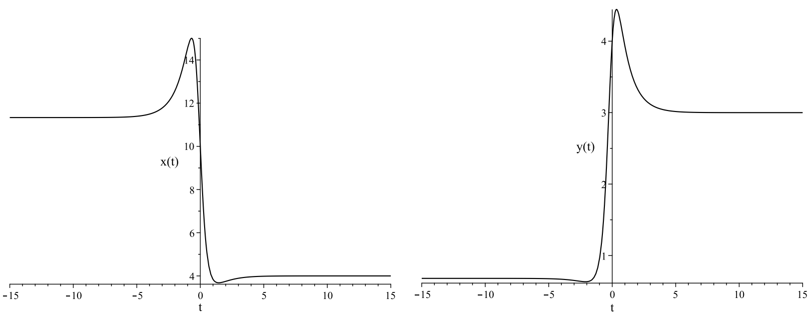
The kink solitary solution to (35) is displayed in Fig. 1.

3.2.2 System of Riccati equations coupled via multiplicative terms

Let us consider the following system of Riccati equations coupled via multiplicative terms:

$$x' = a_0 + a_1 x + a_2 x^2 + a_3 xy; \tag{36}$$

$$y' = b_0 + b_1 y + b_2 y^2 + b_3 xy. \tag{37}$$



(a) (b)

Fig. 2 Dark/bright solitary solutions $x(t)$ (part (a)) and $y(t)$ (part (b)) to (45)-(46) with initial conditions $x(0) = 10$ and $y(0) = 4$.

3.3 Inverse balancing technique

Inverse balancing technique [25] is used in order to obtain necessary and sufficient conditions for the existence of solitary solutions to a system of differential equations as well as to determine the possible order of such solutions. The main idea of this technique is to insert the solitary solution as an anzatz into the considered model which yields a system of linear equations from which the model parameters can be determined in terms of solitary solution parameters. Inverse balancing technique is applied to the system (20)–(23) in Sections 4 and 5.

Note that a direct balancing approach consisting of inserting the solutions of the form (24) into the system (20)–(23) would result in a high-order nonlinear system of algebraic equations with respect to the solution parameters. Direct construction of a solution to this system would not be possible. Due to this, the inverse balancing technique is applied for the analysis of the system (20)–(23).

4 Derivation of necessary and sufficient conditions for the existence of solitary solutions to (20)–(23)

The system of equations (20)–(23) can be transformed via the substitution (25) as follows:

$$\eta\widehat{t}\widehat{x_1}'_{\widehat{t}} = a_{01} + a_{11}\widehat{x_1} + a_{21}\widehat{x_1}^2 + \lambda_1\widehat{x_1}\widehat{y_1}; \quad (47)$$

$$\eta\widehat{t}\widehat{y_1}'_{\widehat{t}} = b_{01} + b_{11}\widehat{y_1} + b_{21}\widehat{y_1}^2 + \mu_1\widehat{x_1}\widehat{y_1} + \gamma_1\left(\widehat{y_2} - \widehat{y_1}\right); \quad (48)$$

$$\eta\widehat{t}\widehat{x_2}'_{\widehat{t}} = a_{02} + a_{12}\widehat{x_2} + a_{22}\widehat{x_2}^2 + \lambda_2\widehat{x_2}\widehat{y_2}; \quad (49)$$

$$\eta\widehat{t}\widehat{y_2}'_{\widehat{t}} = b_{02} + b_{12}\widehat{y_2} + b_{22}\widehat{y_2}^2 + \mu_2\widehat{x_2}\widehat{y_2} + \gamma_2\left(\widehat{y_1} - \widehat{y_2}\right). \quad (50)$$

Let:

$$x_l\left(t\right)=\widehat{x}_l\left(\widehat{t}\right)=\sigma_{1l}\frac{X_l}{T};\quad y_l\left(t\right)=\widehat{y}_l\left(\widehat{t}\right)=\sigma_{2l}\frac{Y_l}{T},\tag{51}$$

$$X_l=X_l\left(\widehat{t}\right)=\left(\widehat{t}-x_{1l}\right)\left(\widehat{t}-x_{2l}\right)\cdots\left(\widehat{t}-x_{nl}\right)=\widehat{t}^n+\chi_{(n-1)l}\widehat{t}^{n-1}+\cdots+\chi_{0l};\tag{52}$$

$$Y_l=Y_l\left(\widehat{t}\right)=\left(\widehat{t}-y_{1l}\right)\left(\widehat{t}-y_{2l}\right)\cdots\left(\widehat{t}-y_{nl}\right)=\widehat{t}^n+\theta_{(n-1)l}\widehat{t}^{n-1}+\cdots+\theta_{0l},\tag{53}$$

$$T=T\left(\widehat{t}\right)=\left(\widehat{t}-t_1\right)\left(\widehat{t}-t_2\right)\cdots\left(\widehat{t}-t_n\right)=\widehat{t}^n+æ_{n-1}\widehat{t}^{n-1}+\cdots+æ_0;\tag{54}$$

$$X_l'=\left(X_l\left(\widehat{t}\right)\right)_{\widehat{t}}',\quad Y_l'=\left(Y_l\left(\widehat{t}\right)\right)_{\widehat{t}}',\quad T'=\left(T\left(\widehat{t}\right)\right)_{\widehat{t}}',\tag{55}$$

where $æ_k, \chi_{kl}, \theta_{kl} \in \mathbb{C}, k = 1, \ldots, n-1, l = 1, 2$. Note that in this paper the order of the solitary solution is defined by the value of the parameter n.

Necessary and sufficient conditions for the existence of solitary solutions to (20)–(23) are further obtained by inserting the solitary solutions (51) into (47)–(50). The system can then be rewritten in the following way:

$$\eta\widehat{t}\sigma_{1l}\frac{X_l'T-X_lT'}{T^2}=a_{0l}+a_{1l}\sigma_{1l}\frac{X_l}{T}+a_{2l}\sigma_{1l}^2\frac{X_l^2}{T^2}+\lambda_l\sigma_{1l}\sigma_{2l}\frac{X_lY_l}{T^2};\tag{56}$$

$$\eta\widehat{t}\sigma_{2l}\frac{Y_l'T-Y_lT'}{T^2}=b_{0l}+c_l\sigma_{2l}\frac{Y_l}{T}+b_{2l}\sigma_{2l}^2\frac{Y_l^2}{T^2}+\mu_l\sigma_{1l}\sigma_{2l}\frac{X_lY_l}{T^2}+\gamma_l\sigma_{2r}\frac{Y_r}{T},\tag{57}$$

where $c_l = b_{1l} - \gamma_l, l, r = 1, 2, r \neq l$.

Multiplying both sides of the equations (56)–(57) by $\frac{T^2}{X_l}$ and T respectively and rearranging the resulting equations yields:

$$\frac{\eta\widehat{t}\sigma_{1l}X_l'T}{X_l}-\frac{a_{0l}T^2}{X_l}=\eta\widehat{t}\sigma_{1l}T'+a_{1l}\sigma_{1l}T+a_{2l}\sigma_{1l}^2X_l+\lambda_l\sigma_{1l}\sigma_{2l}Y_l;\tag{58}$$

$$-\frac{\eta\widehat{t}\sigma_{2l}Y_lT'}{T}-\frac{b_{2l}\sigma_{2l}^2Y_l^2}{T}-\frac{\mu_l\sigma_{1l}\sigma_{2l}X_lY_l}{T}=-\eta\widehat{t}\sigma_{2l}Y_l'+b_{0l}T+c_l\sigma_{2l}Y_l+\gamma_l\sigma_{2r}Y_r.\tag{59}$$

Note that all terms on the right-hand side of the equations (58)–(59) are of the order n; numerators on the left-hand side of the equations (58)–(59) are of the order $2n$; denominators on the left-hand side of the equations (58)–(59) are of the order n. Thus, in order for the equations (58)–(59) to hold true, the denominators must be cancelled out, i.e. the following conditions must be satisfied:

$$\eta\widehat{t}\sigma_{1l}X_l'-a_{0l}T=\sigma_{1l}\alpha_lX_l;\tag{60}$$

$$-\eta\widehat{t}\sigma_{2l}T'-b_{2l}\sigma_{2l}^2Y_l-\mu_l\sigma_{1l}\sigma_{2l}X_l=\sigma_{2l}\beta_lT,\tag{61}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}\backslash\{0\}$ are arbitrary constants.

Consequently, conditions (60)–(61) are necessary for the existence of solitary solutions (51) in the system (20)–(23). Note that conditions (60)–(61) can be rearranged as follows:

$$a_{0l}T + \alpha_l\sigma_{1l}X_l - \eta\widehat{t}\sigma_{1l}X_l' = 0; \tag{62}$$

$$\eta\widehat{t}T' + \beta_l T + \mu_l\sigma_{1l}X_l + b_{2l}\sigma_{2l}Y_l = 0, \tag{63}$$

where $l = 1, 2$.

Inserting (60)–(61) into (58)–(59) yields the following system of algebraic equations:

$$\alpha_l T = \eta\widehat{t}T' + a_{1l}T + a_{2l}\sigma_{1l}X_l + \lambda_l\sigma_{2l}Y_l; \tag{64}$$

$$\sigma_{2l}\beta_l Y_l = -\eta\widehat{t}\sigma_{2l}Y_l' + b_{0l}T + c_l\sigma_{2l}Y_l + \gamma_l\sigma_{2r}Y_r, \tag{65}$$

where $l, r = 1, 2; l \neq r$.

Equations (64) - (65) correspond to the sufficient conditions for the existence of solitary solutions (51) in the system (20)–(23). Note that those conditions can be rearranged as follows:

$$\eta\widehat{t}T' + (a_{1l} - \alpha_l)T + a_{2l}\sigma_{1l}X_l + \lambda_l\sigma_{2l}Y_l = 0; \tag{66}$$

$$\gamma_l\sigma_{2r}Y_r + (c_l - \beta_l)\sigma_{2l}Y_l + b_{0l}T - \eta\sigma_{2l}\widehat{t}Y_l' = 0. \tag{67}$$

Lemma 1 *Solitary solutions (51) satisfy the system (20)–(23) if and only if the conditions (62)-(63) and (66)-(67) hold true.*

5 Determination of the maximal order of the solitary solution (51) to (20) - (23)

In this subsection the inverse balancing technique (see Section 3.3) is applied in order to express the coefficients of the system (20) - (23) in terms of the solitary solution (51) parameters as well as to determine the maximal order of the solitary solution (51) to (20) - (23). Consider the following one-to-one mappings:

$$T \mapsto \overrightarrow{T} = (1, æ_{n-1}, \ldots, æ_0); \tag{68}$$

$$X_l \mapsto \overrightarrow{X_l} = \left(1, \chi_{(n-1)l}, \ldots, \chi_{0l}\right); \tag{69}$$

$$Y_l \mapsto \overrightarrow{Y_l} = \left(1, \theta_{(n-1)l}, \ldots, \theta_{0l}\right); \tag{70}$$

$$\widehat{t}T' \mapsto \overrightarrow{\left(\widehat{t}T'\right)} = \left(n, (n-1)æ_{n-1}, \ldots, æ_1, 0\right); \tag{71}$$

$$\widehat{t}X_l' \mapsto \overrightarrow{\left(\widehat{t}T'\right)} = \left(n, (n-1)\chi_{(n-1)l}, \ldots, \chi_{1l}, 0\right); \tag{72}$$

$$\widehat{t}Y_l' \mapsto \overrightarrow{\left(\widehat{t}T'\right)} = \left(n, (n-1)\theta_{(n-1)l}, \ldots, \theta_{1l}, 0\right), \tag{73}$$

where $l = 1, 2$. Note that in this case solutions L_{kl} and N_{kl} $(k = 1, 2, 3)$ are not necessarily equal.

However, when $n \geq 2$, systems (86) and (89) can only have a single unique solution:

$$\begin{aligned} L_{1l} = N_{1l} &= \frac{(p_{1l}n - 1)h_{0l} - p_{0l}(nh_{1l} - 1)}{(1 - p_{1l})h_{0l} + (h_{1l} - 1)p_{0l} + (p_{1l} - h_{1l})}, \\ L_{2l} = N_{2l} &= \frac{(1 - n)h_{0l} - 1 + nh_{1l}}{(1 - p_{1l})h_{0l} + (h_{1l} - 1)p_{0l} + (p_{1l} - h_{1l})}, \\ L_{3l} = N_{2l} &= \frac{(n - 1)p_{0l} + 1 - np_{1l}}{(1 - p_{1l})h_{0l} + (h_{1l} - 1)p_{0l} + (p_{1l} - h_{1l})} \end{aligned} \tag{92}$$

if and only if the following conditions with respect to solitary solution (51) parameters hold true:

$$k + L_{1l} + L_{2l}p_{kl} + L_{3l}h_{kl} = 0, \quad k = 2, \ldots, n - 1; \tag{93}$$

$$(1 - p_{1l})h_{0l} + (-1 + h_{1l})p_{0l} + p_{1l} - h_{1l} \neq 0, \tag{94}$$

where $l = 1, 2$.

Note that in this case, applying $L_{kl} = N_{kl}$ $(k = 1, 2, 3)$ to (78)-(81) yields:

$$a_{2l} = \mu_l; \quad b_{2l} = \lambda_l, \tag{95}$$

where $l = 1, 2$.

Applying (68), (70) and (73) to (77) results in the following system of linear equations with respect to K_{1l}, K_{2l} and K_{3l}:

$$\begin{cases} 1 + K_{1l} + K_{2l} + nK_{3l} = 0; \\ h_{kr} + K_{1l}h_{kl} + K_{2l} + K_{3l}kh_{kl} = 0, \quad k = n - 1, \ldots, 0, \end{cases} \tag{96}$$

where $l, r = 1, 2$, $l \neq r$.

The system (96) yields the expressions for K_{1l}, K_{2l}:

$$K_{1l} = \frac{h_{0r} - nK_{3l} - 1}{1 - h_{0l}}, \quad K_{2l} = \frac{(nK_{3l} + 1)h_{0l} - h_{0r}}{1 - h_{0l}}, \tag{97}$$

and the following conditions:

$$h_{kr} + K_{1l}h_{kl} + K_{2l} + K_{3l}kh_{kl} = 0 \quad k = 1, \ldots, n - 1; \tag{98}$$

where $l, r = 1, 2$, $l \neq r$.

Thus, necessary and sufficient conditions for the existence of solitary solutions (51) to the system (20)–(23) can be reformulated in terms of solitary solution parameters as follows:

Lemma 2 *Solitary solutions (51) satisfy the system (20)–(23) if and only if the conditions (93) and (98) hold true.*

Note that applying (85) and (87) to (51) yields the following expression of the solitary solution:

$$x_l(t) = \widehat{x_l}\left(\widehat{t}\right) = \sigma_{1l}\frac{X_l}{T} = \sigma_{1l}\frac{\left(\widehat{t}\right)^n + p_{(n-1)l}æ_{n-1}\left(\widehat{t}\right)^{n-1} + \cdots + p_{0l}æ_0}{\left(\widehat{t}\right)^n + æ_{n-1}\left(\widehat{t}\right)^{n-1} + \cdots + æ_0}; \tag{99}$$

$$y_l(t) = \widehat{y_l}\left(\widehat{t}\right) = \sigma_{2l}\frac{Y_l}{T} = \sigma_{2l}\frac{\left(\widehat{t}\right)^n + h_{(n-1)l}æ_{n-1}\left(\widehat{t}\right)^{n-1} + \cdots + h_{0l}æ_0}{\left(\widehat{t}\right)^n + æ_{n-1}\left(\widehat{t}\right)^{n-1} + \cdots + æ_0}, \tag{100}$$

for $l = 1, 2$.

Algebraically solving the system of necessary and sufficient conditions defined in Lemma 2 for various values of n yields the conclusion, summarized in the Lemma below.

Lemma 3 *System of differential equations (20)–(23) can admit solitary solutions of any order $n \in \mathbb{N}$. However, two cases are present:*

- *If $n \leq 3$, system of necessary and sufficient conditions defined in Lemma 2 can be solved without additional constraints on solitary solution parameters. Moreover, selecting different values of $æ_k, k = 0, \ldots, n-1$ in (99)-(100) generates an infinite number of solitary solutions corresponding to a single system of differential equations (20) - (23). Note that in case of $n = 2, 3$, constraints (95) on differential equation parameters must be satisfied in order to ensure the existence of the solitary solution, whereas for $n = 1$ these constraints are unnecessary.*
- *If $n > 3$, system of necessary and sufficient conditions defined in Lemma 2 can be solved if any $(n-3)$ parameters $æ_k, k \in \{0, \ldots, n-1\}$ are equal to zero. Then, selecting different values of remaining parameters $æ_k$ in (99)-(100) generates an infinite number of solitary solutions corresponding to a single system of differential equations (20) - (23). Moreover, constraints (95) on differential equation parameters must be satisfied in order to ensure the existence of the solitary solution.*

The auxiliary parameters A_{kl}, L_{ml}, N_{ml} and K_{ml} form an essential link between the parameters of the system (20)–(23) and the solitary solution (51). If it is possible to determine the auxiliary parameters from the solitary solution, the system parameters can be computed via (82), (83). Conversely, if the system parameters are known they can be used to determined auxiliary parameters, which in turn yield the solitary solution parameters via (74)–(81).

6 Computational experiments. Third order solitary solutions to (20)-(23)

In this subsection Lemma 2 is applied in order to show that system (20)–(23) can admit third order ($n = 3$) solitary solutions:

$$x_1(t) = \widehat{x_1}\left(\widehat{t}\right) = \sigma_{11}\frac{X_1}{T} = \sigma_{11}\frac{\left(\widehat{t}\right)^3 + p_{21}æ_2\left(\widehat{t}\right)^2 + p_{11}æ_1\widehat{t} + p_{01}æ_0}{\left(\widehat{t}\right)^3 + æ_2\left(\widehat{t}\right)^2 + æ_1\widehat{t} + æ_0}; \quad (101)$$

$$x_2(t) = \widehat{x_2}\left(\widehat{t}\right) = \sigma_{12}\frac{X_2}{T} = \sigma_{12}\frac{\left(\widehat{t}\right)^3 + p_{22}æ_2\left(\widehat{t}\right)^2 + p_{12}æ_1\widehat{t} + p_{02}æ_0}{\left(\widehat{t}\right)^3 + æ_2\left(\widehat{t}\right)^2 + æ_1\widehat{t} + æ_0}; \quad (102)$$

$$y_1(t) = \widehat{y_1}\left(\widehat{t}\right) = \sigma_{21}\frac{Y_1}{T} = \sigma_{21}\frac{\left(\widehat{t}\right)^3 + h_{21}æ_2\left(\widehat{t}\right)^2 + h_{11}æ_1\widehat{t} + h_{01}æ_0}{\left(\widehat{t}\right)^3 + æ_2\left(\widehat{t}\right)^2 + æ_1\widehat{t} + æ_0}; \quad (103)$$

$$y_2(t) = \widehat{y_2}\left(\widehat{t}\right) = \sigma_{22}\frac{Y_2}{T} = \sigma_{22}\frac{\left(\widehat{t}\right)^3 + h_{22}æ_2\left(\widehat{t}\right)^2 + h_{12}æ_1\widehat{t} + h_{02}æ_0}{\left(\widehat{t}\right)^3 + æ_2\left(\widehat{t}\right)^2 + æ_1\widehat{t} + æ_0}. \quad (104)$$

As shown in Section 5, solitary solutions (101)-(104) satisfy the model (20)–(23) if and only if the following conditions hold true:

$$2 + L_{1l} + L_{2l}p_{2l} + L_{3l}h_{2l} = 0; \quad (105)$$

$$h_{1r} + K_{1l}h_{1l} + K_{2l} + K_{3l}h_{1l} = 0; \quad (106)$$

$$h_{2r} + K_{1l}h_{2l} + K_{2l} + 2K_{3l}h_{2l} = 0, \quad (107)$$

where

$$L_{1l} = \frac{(3p_{1l} - 1)h_{0l} - p_{0l}(3h_{1l} - 1)}{(1 - p_{1l})h_{0l} + (h_{1l} - 1)p_{0l} + (p_{1l} - h_{1l})}; \quad (108)$$

$$L_{2l} = \frac{-2h_{0l} - 1 + 3h_{1l}}{(1 - p_{1l})h_{0l} + (h_{1l} - 1)p_{0l} + (p_{1l} - h_{1l})}; \quad (109)$$

$$L_{3l} = \frac{2p_{0l} + 1 - 3p_{1l}}{(1 - p_{1l})h_{0l} + (h_{1l} - 1)p_{0l} + (p_{1l} - h_{1l})}; \quad (110)$$

$$K_{1l} = \frac{h_{0r} - 3K_{3l} - 1}{1 - h_{0l}}; \quad (111)$$

$$K_{2l} = \frac{(3K_{3l} + 1)h_{0l} - h_{0r}}{1 - h_{0l}}; \quad (112)$$

$$p_{kl} = \frac{1}{1 + (3 - k)A_{2l}}; \quad (113)$$

$$A_{2l} = -\frac{1 + A_{1l}}{3}, \quad (114)$$

for $l, r = 1, 2; l \neq r$ and $k = 0, 1, 2$. Note that the system (105)-(107) has 6 equations and 10 unknowns, namely, $K_{31}, K_{32}, A_{11}, A_{12}, h_{01}, h_{02}, h_{11}, h_{12}, h_{21}$ and h_{22}. Thus, 4 unknowns can be chosen freely. Let:

$$A_{11} = \frac{4}{5}; \quad A_{12} = \frac{2}{5}; \quad h_{12} = \frac{139919}{7619}; \quad h_{22} = \frac{39493}{15238}. \tag{115}$$

Then, solving (105)-(107) with respect to $K_{31}, K_{32}, h_{01}, h_{02}, h_{11}, h_{21}$ yields:

$$\begin{aligned} &K_{31} = -\frac{7834855}{274284}; \quad K_{32} = -\frac{685710}{223853}; \quad h_{01} = -\frac{34171}{31979}; \\ &h_{02} = -\frac{10021}{7619}; \quad h_{11} = -\frac{170881}{31979}; \quad h_{21} = \frac{89309}{31979}. \end{aligned} \tag{116}$$

Using (115) and (116), parameters (108)-(114) can be evaluated as follows:

$$\begin{aligned} &L_{11} = -\frac{9929}{4410}; \quad L_{12} = -\frac{12029}{4410}; \quad L_{21} = -8; \quad L_{22} = -2; \\ &L_{31} = \frac{31979}{4410}; \quad L_{32} = \frac{7619}{4410}; \quad K_{11} = \frac{199005317}{4937112}; \quad K_{12} = \frac{33736932}{10968797}; \\ &K_{21} = \frac{219139741}{4937112}; \quad K_{22} = \frac{56093641}{10968797}; \quad p_{01} = -\frac{5}{4}; \quad p_{02} = -\frac{5}{2}; \\ &p_{11} = -5; \quad p_{12} = 15; \quad p_{21} = \frac{5}{2}; \quad p_{22} = \frac{15}{8}; \quad A_{21} = -\frac{3}{5}; \quad A_{22} = -\frac{7}{15}. \end{aligned} \tag{117}$$

Since parameters (115) and (116) ensure the validity of conditions (105)-(107), solitary solution (101)-(102) parameters $æ_0, æ_1, æ_2, \sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22}, \eta, t_0$ can be chosen freely. Moreover, selecting different values of $æ_0, æ_1, æ_2$ generates an infinite number of third order solitary solutions corresponding to a single system of differential equations (20) - (23). Consider the case:

$$\begin{aligned} &æ_0 = 8; \quad æ_1 = 14; \quad æ_2 = 7; \quad \eta = 4; \quad t_0 = -5; \\ &\sigma_{11} = \frac{3}{5}; \quad \sigma_{12} = \frac{7}{15}; \quad \sigma_{21} = \frac{7834855}{274284}; \quad \sigma_{22} = \frac{685710}{223853}. \end{aligned} \tag{118}$$

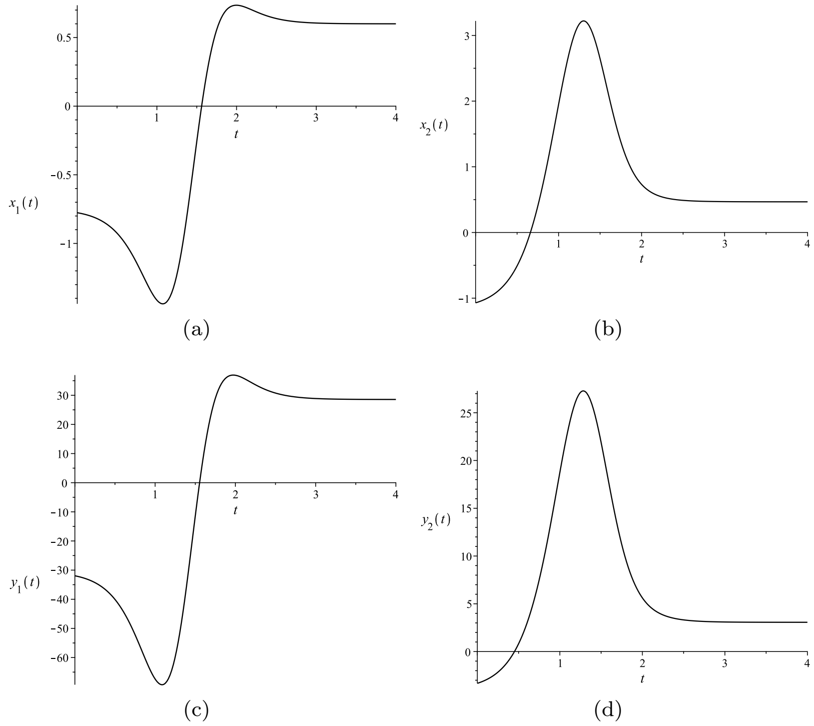


Fig. 3 Third order solitary solutions $x_1(t)$ (part (a)), $x_2(t)$ (part (b)), $y_1(t)$ (part (c)) and $y_2(t)$ (part (d)) to (123)-(126).

separatrices in the paradigmatic Hodgkin-Huxley model [39]. A control technique based on small impulses for silencing a random network of such neurons is proposed in [39, 11].

It was demonstrated in this paper that solitary solutions of an arbitrary order do exist in the diffusively coupled Lotka-Volterra systems. Necessary and sufficient conditions for the existence of such solutions were derived in terms of the system and solution parameters using the inverse balancing technique.

Finding soliton-type solutions to the meta-model of coupled Lotka-Volterra systems would allow to identify the structure of separatrices in the 4-dimensional phase space. The knowledge of the system of separatrices would serve for designing algorithms for the control of transient processes. As discussed in the previous sub-sections, the meta-model of coupled Lotka-Volterra systems has some connections with the phenomenological model of metastasis and the SEIR COVID-19 model. Tuning these connections and designing algorithms for the control of transient processes remains a definite objective of future research.

Declarations

Availability of data and material

No data was used to obtain the results of this manuscript.

Competing interests

The authors declare no competing interests.

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Authors' contributions

I. Timofejeva wrote the manuscript text, performed the necessary symbolic computations. T. Telksnys, Z. Navickas and M. Ragulskis developed the methodology used in the derivation of necessary and sufficient conditions for the existence of solitary solutions. R. Marcinkevicius performed the necessary numerical computation and validated the results using numerical algorithms.

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References

1. Abbasbandy, S., Parkes, E.: Solitary-wave solutions of the degasperis–procesi equation by means of the homotopy analysis method. International Journal of Computer Mathematics **87**(10), 2303–2313 (2010)
2. Ablowitz, M.J., Segur, H.: Solitons and the inverse scattering transform. SIAM (1981)
3. Aerts, D., Czachor, M., Kuna, M., Sozzo, S.: Systems, environments, and soliton rate equations: A non-kolmogorovian framework for population dynamics. Ecological modelling **267**, 80–92 (2013)
4. Annas, S., Pratama, M.I., Rifandi, M., Sanusi, W., Side, S.: Stability analysis and numerical simulation of seir model for pandemic covid-19 spread in indonesia. Chaos, Solitons & Fractals **139**, 110072 (2020)
5. Bomze, I.M.: Lotka-volterra equation and replicator dynamics: a two-dimensional classification. Biological cybernetics **48**(3), 201–211 (1983)
6. Bomze, I.M.: Lotka-volterra equation and replicator dynamics: new issues in classification. Biological cybernetics **72**(5), 447–453 (1995)
7. Bratus, A., Samokhin, I., Yegorov, I., Yurchenko, D.: Maximization of viability time in a mathematical model of cancer therapy. Mathematical biosciences **294**, 110–119 (2017)

8. Cole, D.C., Lamb, E.S., DelHaye, P., Diddams, S.A., Papp, S.B.: Soliton crystals in kerr resonators. Nature Photonics **11**(10), 671–676 (2017)
9. Cooke, K.L., Van Den Driessche, P.: Analysis of an seirs epidemic model with two delays. Journal of Mathematical Biology **35**(2), 240–260 (1996)
10. Doyon, B., Yoshimura, T., Caux, J.S.: Soliton gases and generalized hydrodynamics. Physical review letters **120**(4), 045301 (2018)
11. Fedaravičius, A.P., Cao, M., Ragulskis, M.: Control of a dendritic neuron driven by a phase-independent stimulation. Chaos, Solitons & Fractals **85**, 77–83 (2016)
12. Fermi, E., Pasta, P., Ulam, S., Tsingou, M.: Studies of the nonlinear problems. Tech. rep., Los Alamos Scientific Lab., N. Mex. (1955)
13. Hayakawa, M., Hiraiwa, T., Wada, Y., Kuwayama, H., Shibata, T.: Polar pattern formation induced by contact following locomotion in a multicellular system. Elife **9**, e53609 (2020)
14. Hethcote, H.W., Van den Driessche, P.: Some epidemiological models with nonlinear incidence. Journal of Mathematical Biology **29**(3), 271–287 (1991)
15. Khyar, O., Allali, K.: Global dynamics of a multi-strain seir epidemic model with general incidence rates: application to covid-19 pandemic. Nonlinear Dynamics pp. 1–21 (2020)
16. Kordeweg, D., de Vries, G.: On the change of form of long waves advancing in a rectangular channel, and a new type of long stationary wave. Phil. Mag **39**, 422–443 (1895)
17. Kudryashov, N.A.: Exact solitary waves of the fisher equation. Physics Letters A **342**(1-2), 99–106 (2005)
18. Kudryashov, N.A.: A note on the G'/G-expansion method. Applied Mathematics and Computation **217**(4), 1755–1758 (2010)
19. Kudryashov, N.A., Loguinova, N.B.: Extended simplest equation method for nonlinear differential equations. Applied Mathematics and Computation **205**(1), 396–402 (2008)
20. Liu, Z., Yang, C.: A mathematical model of cancer treatment by radiotherapy followed by chemotherapy. Mathematics and Computers in Simulation **124**, 1–15 (2016)
21. McKendrick, A., Pai, M.K.: Xlv.the rate of multiplication of micro-organisms: a mathematical study. Proceedings of the Royal Society of Edinburgh **31**, 649–653 (1912)
22. Nagatani, T.: Migration difference in diffusively-coupled prey–predator system on heterogeneous graphs. Physica A: Statistical Mechanics and its Applications **537**, 122705 (2020)
23. Navickas, Z., Bikulciene, L., Rahula, M., Ragulskis, M.: Algebraic operator method for the construction of solitary solutions to nonlinear differential equations. Commun Nonlinear Sci Numer Simul **18**, 1374–1389 (2013)
24. Navickas, Z., Marcinkevicius, R., Telksnys, T., Ragulskis, M.: Existence of second order solitary solutions to riccati differential equations coupled with a multiplicative term. IMA Journal of Applied Mathematics **81**(6), 1163–1190 (2016)
25. Navickas, Z., Ragulskis, M., Telksnys, T.: Existence of solitary solutions in a class of nonlinear differential equations with polynomial nonlinearity. Appl Math Comput **283**, 333–338 (2016)
26. Navickas, Z., Telksnys, T., Ragulskis, M.: Comments on the exp-function method and generalized solitary solutions. Computers & Mathematics with Applications **69**(8), 798–803 (2015)
27. Navickas, Z., Vilkas, R., Telksnys, T., Ragulskis, M.: Direct and inverse relationships between riccati systems coupled with multiplicative terms. Journal of biological dynamics **10**(1), 297–313 (2016)
28. Pikovsky, A., Kurths, J., Rosenblum, M., Kurths, J.: Synchronization: a universal concept in nonlinear sciences, vol. 12. Cambridge university press (2003)
29. Polyanin, A.D., Zaitsev, V.F.: Handbook of nonlinear partial differential equations. CRC press (2004)
30. Rezazadeh, H., Korkmaz, A., Khater, M.M., Eslami, M., Lu, D., Attia, R.A.: New exact traveling wave solutions of biological population model via the extended rational sinh-cosh method and the modified khater method. Modern Physics Letters B **33**(28), 1950338 (2019)
31. Rhodes, A., Hillen, T.: A mathematical model for the immune-mediated theory of metastasis. Journal of theoretical biology **482**, 109999 (2019)
32. Rogers, C., Rogers, C., Schief, W.: Bäcklund and Darboux transformations: geometry and modern applications in soliton theory, vol. 30. Cambridge University Press (2002)

33. Rogers, C., Shadwick, W.F.: Bäcklund transformations and their applications. Academic press (1982)
34. Russell, J.S.: Report on Waves: Made to the Meetings of the British Association in 1842-43 (1845)
35. Saito, M.M., Imoto, S., Yamaguchi, R., Sato, H., Nakada, H., Kami, M., Miyano, S., Higuchi, T.: Extension and verification of the seir model on the 2009 influenza a (h1n1) pandemic in japan. Mathematical biosciences **246**(1), 47–54 (2013)
36. Scott, A. (ed.): Encyclopedia of Nonlinear Science. Routledge, New York (2004)
37. Scott, A.: Encyclopedia of nonlinear science. Routledge (2006)
38. Sigal, D., Przedborski, M., Sivaloganathan, D., Kohandel, M.: Mathematical modelling of cancer stem cell-targeted immunotherapy. Mathematical Biosciences **318**, 108269 (2019)
39. Telksnys, T., Navickas, Z., Timofejeva, I., Marcinkevicius, R., Ragulskis, M.: Symmetry breaking in solitary solutions to the hodgkin–huxley model. Nonlinear Dynamics **97**(1), 571–582 (2019)
40. Verhulst, P.F.: Notice sur la loi que la population suit dans son accroissement. Corresp. Math. Phys. **10**, 113–126 (1838)
41. Wang, M., Li, X., Zhang, J.: The (G'/G)-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics. Physics Letters A **372**(4), 417–423 (2008)
42. Zabusky, N.J., Kruskal, M.D.: Interaction of" solitons" in a collisionless plasma and the recurrence of initial states. Physical review letters **15**(6), 240 (1965)
43. Zazoua, A., Wang, W.: Analysis of mathematical model of prostate cancer with androgen deprivation therapy. Communications in Nonlinear Science and Numerical Simulation **66**, 41–60 (2019)
44. Zhang, T., Teng, Z.: On a nonautonomous seirs model in epidemiology. Bulletin of mathematical biology **69**(8), 2537–2559 (2007)